SET THEORY WITH A UNIVERSAL SET

By

MICHAEL W. JAMIESON

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Michael W. Jamieson

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Chairman: Dr. Rick L. Smith
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This dissertation presents a new resolution of the set theoretic paradoxes of Russell and Cantor. The work contains a model for a theory of sets which implies that there is a universal set u. In the language of equality and a set membership relation $\cdot E \cdot$, a *strictly quantifier-free* formula is a quantifier-free formula whose atomic subformulas have no variable which occurs more than once. The axioms of the theory are extensionality and the comprehension scheme for strictly quantifier-free formulas. Any model of this theory satisfies the sentences "the universal set u exists" and "the graphs of the entire membership and identity relations over u exist". The richest model presented is also *internally countable*, i.e. it satisfies the sentences "The ordinal ω exists" and "There is a function $f : u \xrightarrow{1-1} \omega$". The author defines a list of *Gödel-like* operations. This list contains a variant appropriate to the context of a universal set for each operation used in construction of Gödel's L from the ordinals, except domain formation and union of an arbitrary set of sets. Gödel-like closure implies strictly quantifier-free comprehension.

This model is constructed inside a countable atomic Boolean algebra as a direct limit of S-algebras, i.e. atomic subalgebras $C_n$with one-to-one maps $\mathrm{S}_n : C_n \xrightarrow{1-1} C_n$ such that for all $x \in C_n$ the element $\mathrm{S}_n(x)$ is an atom of C_n. These maps $\mathrm{S}_n(\cdot)$ induce membership relations $\cdot E_n \cdot$ by $x E_n y \leftrightarrow \mathrm{S}_n(x) \leq y$. The structures $\langle C_n, E_n \rangle$ satisfy the sentences "$\mathrm{S}_n(x)$ is the singleton of x" and "the universe is closed under binary union, intersection and complement". Gödel-like closure and internal countability are preserved in an extensional substructure which is constructed by iteratively applying the fact that every countable Boolean algebra is projective.

CHAPTER 1
INTRODUCTION AND SURVEY

This dissertation presents a model of the language of a membership relation $\cdot E \cdot$ with the following properties.

(i) The model satisfies a comprehension axiom scheme which applies to those quantifier-free formulas with arbitrary parameters which contain no atomic subformula vRv, i.e. no atomic subformula in which a variable is repeated.

(ii) Equality in the model is determined by the principle of extensionality

(iii) The model satisfies closure under power set formation.

(iv) The model satisfies the statement "ω exists".

(v) The model satisfies an axiom of internal countability.

Russell observed a contradiction in the original Frege set theory. If r is the class $\{x | x \notin x\}$, then $r \in r \leftrightarrow r \notin r$. This argument uses the comprehension axiom for the formula $x \not\in x$, which contains the subformula xEx and is therefore excluded from the comprehension scheme described in (i). The scheme here differs from the original inconsistent scheme in two ways. First, this scheme excludes instances for quantified formulas, since quantification is beyond the scope of this work. Second, this scheme also excludes instances with a variable repeated in an atomic subformula. The rationale for this exclusion is that such formulas may be regarded as containing an implicit quantification. For example, the formula $x \not\in x$ is logically equivalent to $\exists y[x \not\in y \wedge x = y]$. This rationale is discussed later in the chapter.

DEFN 1.1. *Strictly quantifier-free formulas* in $\mathcal{L}(E, =)$ are those quantifier-free formulas with no subformula vEv or $v = v$ for any variable v. *Strictly quantifier-free*

comprehension with parameters for $\mathcal{L}(E,=)$ is the axiom scheme which contains, for every strictly quantifier-free formula $F(\vec{p},\vec{x})$, the axiom $\forall\vec{p}\exists y\forall\vec{x}[xEy \leftrightarrow F(\vec{p},\vec{x})]$. Note that strictly quantifier-free comprehension allows variable repetition in distinct atomic subformulas, for example $x \in y \wedge y \in x$.

Chapter 4 contains an extensional model $\langle B, E\rangle$ of strictly quantifier-free comprehension. In such models one can define the Boolean algebra operations. The axioms of Boolean algebras are derivable in the theory strictly quantifier-free comprehension plus extensionality. Operations similar to most of those which Gödel used to define the class L of constructible sets are also definable, and the model in Chapter 4 is closed under these operations. Any structure closed under these operations satisfies the scheme strictly quantifier-free comprehension. The proof follows a theorem by Gödel about the structure L of constructible sets [1].

The model in Chapter 4 satisfies the additional axiom that there is a one-to-one function f in the universe which embeds the universe into ω.

DEFN 1.2. A structure $\langle C, E, = \rangle$ is *internally countable* if it satisfies the following:

(i) There is a universal set u, i.e. $\forall x[xEu]$

(ii) There is an element ω whose elements with respect to E form a structure isomorphic to the natural numbers.

(iii) The structure $\langle C, E, = \rangle$ satisfies $\exists f[f : u \xrightarrow{1-1} \omega]$.

Compared to internal countability, Russell's paradox is a somewhat simpler issue, so it is considered first. However, the possibility of error in the ZF treatment of uncountability is much more important.

A model of strictly quantifier-free comprehension contains a universal set u, the membership relation e and the identity relation i as elements. This theory implies

that $\sim e \cap i = \{(x,y) | \, xEy \wedge x = y\} = \{(x,x) | \, xEx\}$ is an element of the structure. The formula $xEy \wedge yEx$, with variable repetitions, has an extension $e \cap e^{-1}$ which can be constructed with relation inverses and intersection. The idea of strictly quantifier-free comprehension is that to construct the extension of xEx requires a distinctly different capability. It requires collection of the fixed points of a relation. This is an intuitively distinct and more complicated kind of task. The distinction is analogous to the difference between $y = Ax$, which represents the simple matrix multiplication relation, and $x = Ax$, which represents collection of the eigenvectors whose eigenvalue is 1.

A formal distinction between these two kinds of tasks is evident in the theory of cylindric algebras [2], also known as quantified Boolean algebras. A cylindric algebra is a Boolean algebra with operations c_α, which are called cylindrifications, and elements $d_{\alpha,\beta}$, which are called diagonalization elements. Elements of the algebra correspond to formulas in a first order language. A cylindrification operation c_α corresponds to quantification over a variable x_α. The diagonalization elements correspond to atomic formulas $x_\alpha = x_\beta$.

The cylindrification operations by themselves do not extend Boolean algebras sufficiently to represent the predicate calculus. Indeed, it is the diagonal arguments which require the additional diagonalization elements $d_{\alpha,\beta}$. In a cylindric algebra, an atomic formula $\mathrm{p}(x_1, x_2, x_3)$ is associated with a generator g_p. The formula $\mathrm{p}(x_1, x_2, x_1)$, with variable repetition, is not generally associated with a distinct generator. Such redundancy can be allowed, but it produces non-extensionality. Instead, the atomic formula $\mathrm{p}(x_1, x_2, x_1)$ can be generally associated with the natural representation of the logically equivalent formula $\exists x_3[\mathrm{p}(x_1, x_2, x_3) \wedge x_1 = x_3]$. That representation is $\mathrm{c}_3(g_\mathrm{p} \wedge d_{1,3})$. Now the representation of the atomic formula with a repeated variable does involve a quantifier.

Thus, Russell's construction is done without apparent quantification, because of an incidental feature of our particular formulation of first order predicate calculus. In this formulation, relation construction and fixed point collection appear so similar that the set theory's naive comprehension axioms did not distinguish them. Suppose a well-formed first order atomic formula is defined as a predicate symbol p followed by a variable x_1, and another (i.e. a distinct) variable x_2, and another variable x_3 and so on, until we have the number of variables appropriate for p. Then $x \not\in x$ simply is not a well-formed formula.

While this may not be natural in the formalism of first order predicate calculus, the corresponding distinction does arise naturally in cylindric algebras. This can happen because the equivalence of first order predicate calculus with the theory of cylindric algebras is not required to preserve naturality. Two points favor cylindric algebras in this issue. One is the inconsistency which results when one accepts xEx as an atomic formula. The other is the intuition about the distinction with concrete examples, like $y = Ax$ and $x = Ax$. This work begins development of the view that, as a consequence of suppressing this distinction, the presentation of ideal set theory simply said something unintended. In fact, it happened to say something false, something provably false (hence, an outright inconsistency).

Constructions similar to Russell's occur in arguments based on uncountable infinities. These arguments contain certain statements which are similar to Russell's axiom of comprehension. It is possible to salvage these statements, which this work takes to be misstatements, since these statements do not contradict themselves. Still, there remains the question whether they contradict the truth. Russell's and Cantor's constructions are very similar. Indeed, Russell's paradox was intended to distill the essence of Cantor's diagonal arguments. Consideration of Russell's set is the natural

initial point of attack, because it is a simpler construction with a much stronger result.

It is desirable for set theory to treat these paradoxes much the same. Chapters 2 and 4 contain models $\langle C, E, = \rangle$ of internal countability. So models exist for the quantifier-free part of the construction of the paradoxical elements. Therefore this work attributes the contradictions which arise in the traditional analysis of both paradoxes to mistreatment of quantification.

To see how the suggested restrictions relate to the strength of the theory, it is helpful to consider an analogous, more well-understood situation in Second Order Arithmetic. There, all sets contain only natural numbers. Church's Thesis characterizes those arithmetic relations whose definitions give effectively computable decision procedures for them. The effectively decidable relations are those which are definable both by an *existential* formula (a formula with a single existential quantifier) and by a *universal* formula (a formula with a single universal quantifier). In arithmetic, the quantifier-free part of a universal or existential formula is allowed to refer to addition, multiplication, equality of natural numbers and membership $n \in A$. Thus, the class of effective relations is indeed much more limited than the full class of definable relations.

A quantifier-free arithmetic definition is effectively decidable, relative to whatever set parameters occur in the formula. Equality of natural numbers is defined by $m = n$. So equality of natural numbers is effectively decidable. Equality of two sets of natural numbers is defined by $x = y \leftrightarrow \forall z[z \in x \leftrightarrow z \in y]$, and it is not generally definable with a single existential quantifier. So it is not effectively decidable, and reasonable models of set theory need not contain it.

In second order arithmetic, equality of sets of natural numbers is roughly of the complexity of the halting problem relative to the elements x and y. Second

order arithmetic is interpretable as a two-sorted first order theory where equality over one sort, the natural numbers, is simple. General equality will have at least the complexity of the equality of sets of natural numbers. By analogy, in set theory with a universal set, introduction of explicit reference to general equality in Russell's paradox raises the complexity of that construction.

Strictly quantifier-free comprehension calls for an element e which witnesses the comprehension axiom yEz, and an element i which is (a code for) the identity relation. This theory implies closure under Boolean operations as used in the construction of Russell's class. How can such a theory be consistent? The element $\sim e \wedge i$ witnesses the comprehension axiom for the strictly quantifier-free formula $y \not\! E z \wedge y = z$, but not for the more complex formula $\exists y[y \not\! E z \wedge y = z]$, nor for its equivalent formula $z \not\! E z$. Note that both of those formulas contain only one free variable. If $p = \sim e \wedge i$, then $p \not\! E p$ simply because p is not an ordered pair.

One part of the contradiction becomes more complicated, but it is does retain some consequence. With the usual Wiener-Kuratowski codes for ordered pairs, i.e. $(y,z) = \{\{y\},\{y,z\}\}$, it follows that

(1-1)
$$\begin{aligned} pEp &\longrightarrow p = (q,q), && \text{where } q \notin q \\ &\longrightarrow p = \{\{q\}\}, && \text{where } q \notin q \\ &\longrightarrow \{q\} = p = \{\{q\}\}, \text{ and } q \notin q && \text{since } pEp \\ &\longrightarrow q = \{q\}, \text{ and } q \notin q, \text{ a contradiction} && \text{since } \{\cdot\} \text{ is 1-1} \end{aligned}$$

So with the usual codes for ordered pairs, one has $p \notin p$. This is not problematic. Any interesting model assigns infinitely many elements to $\sim e \wedge i$. Then p cannot be an ordered pair in the usual sense, and the second half of Russell's argument is

preempted. Regardless of the ordered pair coding, the range of p would be Russell's set. Thus, Russell's paradox implies that not every relation has a range.

In ZF the axiom scheme of replacement says that every formula which defines a single-valued relation with a set as its domain has a set as its range. That axiom was not a part of Zermelo's set theory. It was added later by Fraenkel [3]. Thus the prospect of a foundational theory without the full replacement scheme is not unreasonable.

Note that the restriction on variable repetition does not change the notion of definability of a relation. Indeed, even if one preempts the comprehension axioms for formulas with variable repetition in an atomic subformula by defining the class of well-formed formulas to exclude such formulas, they are replaced by other formulas in the language. For any such formula excluded from this modified language, the language includes one which is logically equivalent to it. In the case of Russell's set, there is the formula $\exists y[xEy \wedge x = y]$. Either way, this restriction of the comprehension scheme does change the way one computes the quantifier complexity of a relation. In particular, it changes the effective decidability of the relation relative to the primitives of a language.

So strictly quantifier-free comprehension dispenses with those quantifier-free formulas which are problematic, such as Russell's, and gives a simple treatment to all other quantifier-free formulas to open the way for treatment of formulas with quantifiers. Difficulties must then arise with the same paradoxical sets defined by formulas with quantifiers. In that context, the paradoxes say something about correct treatment of quantifiers within limits imposed by the simpler, more well-understood aspects of logical construction.

Before surveying of the field of set theory with a universal set, consider why a set theory should include a universal set. The reasons include intuition, uniformity

of the theory, a desire for a theory with a standard interpretation and correctness of the theory.

First is the intuition that sets form a Boolean algebra. This is clear from the original acceptance of axioms of the ideal set theory, such as $\forall p \exists y \forall x [x \in y \leftrightarrow x \notin p]$. Also there is an expectation that the structure of sets will reflect the structure of logical formulas. A model of a first order language is a homomorphism $\mathcal{M}$ which interprets variables. The homomorphism maps $\neg A$ to $\sim (A^{\mathcal{M}})$. Chapters 2 and 4 contain models in which, for any set a which might serve as $A^{\mathcal{M}}$, there is a simple unrelativized complement $\sim a$ to serve as $\sim (A^{\mathcal{M}})$, unlike in a model of ZF.

Another aspect of set theory's underlying intuition is the idea that we can formalize most or all of mathematics in terms of *the* membership relation. To represent membership as a class of relations, as required in ZF, is contrary to this intuition. Note that all of the comprehension axioms for $x \in y$ and all other strictly quantifier-free formulas were accepted as intuitively supported in the ideal set theory. They were rejected later because some *other* axioms were inconsistent.

Second, the similarity of basic diagonal arguments, particularly Russell's paradox and Cantor's theorem, suggests a uniform resolution of these paradoxes. Indeed, Russell stated that his paradox arose as an attempt to distill the essence of Cantor's theorem. Russell's paradox demands attention because it is the simpler result and it is so much stronger as to be disastrous.

Chapters 2 and 4 contain models of strictly quantifier-free CA in which the universe also displays internal countability. The models satisfy the sentence "there is an element f which is a one-to-one function and which maps the universe into ω". For the model in Chapter 4, given an arbitrary set a in the model, the element $f \wedge (a \times u)$ is a one-to-one function which maps a into ω. So under this definition of countability, every set in the model is countable.

Third, it would be desirable to have a set theory for which there is a standard model. That is not as clearly true in the case with ZF as with, say, Peano arithmetic. It is easy to find a standard model for Peano Arithmetic, since any first order theory has a prime model, i.e. one which embeds isomorphically into every model. This determines the isomorphism type of the standard model. Even though there are non-standard models of the theory of arithmetic, one understands what it means to ask if Fermat's last theorem is true but not provable. And even though no recursive axiom system decides all questions of arithmetic, a mathematician may use creativity to extend an axiom system with the goal, in principle, of eventually deciding any statement with respect to that one isomorphism type. Now compare the situation with ZF.

For ZF, the class L of constructible sets is a natural minimal model relative to the class of ordinals. Not all logicians want to take it as the standard model, since it lacks much of the interesting structure found in other models of ZF. The purpose of set theory in mathematics is largely to formulate the notion of completeness of the reals, the notion that the line has a point everywhere there can be a point. The reals of L are a small class compared with other, richer models of ZF with, say, $0^{\#}$ or various L-generic reals. So L is not suitable as a standard model and has not been accepted as such. Yet the other models also are also "standard" models.

To some extent, the model V, i.e. the "real world" of ZF, serves as a standard model. Yet the standard model of Peano arithmetic has a known isomorphism type, and this is not the case with V. In Peano arithmetic, if a statement fails only in models with non-standard natural numbers, then it is true in the model that all mathematicians care most about. In ZF, a statement may fail only in models with $0^{\#}$ or only in models without $0^{\#}$. In either case it can remain a viable additional axiom. What assurance is there that one mathematician's vision of V is isomorphic

to another's? As mathematicians apply their creativity to extend ZF, is there any assurance that they will chose the same extension of the theory, as in the case of Peano arithmetic.

Fourth, the universe is an effectively decidable set. Technically, it is not recursive, because it is not a set of natural numbers. However, "always answer 'yes' " is an effective algorithm to correctly decide whether a given object belongs to the universe. The triviality of the algorithm is irrelevant to its correctness and effectiveness. To address the existence of a presented object, one is not required to treat it as an expression and decide if it refers to something which exists. That is not required of a decision procedure for the empty set. "Always answer 'no' " is an acceptable procedure for that problem, whether one considers the effectiveness or recursiveness of the empty set. So the similar algorithm for the universal set does demonstrate its effective decidability.

A clear identification of the source of the paradoxes should not force exclusion of any effectively decidable sets. In particular, a theory with correct resolution of the paradoxes should have models which contain a universal set. More complex classes such as the membership relation are not such obvious matters, though the models in Chapters 2 and 4 of strictly quantifier-free comprehension do contain even the membership relation.

Contrast this with intuitionism and ZF. In ZF, the extension of a simple formula such as $x \not\in a$ is regarded as problematic, and it is described as a proper class. Another well-known way to resolve Russell's paradox is to adopt intuitionistic logic and abandon the Law of Excluded Middle. Set aside the pragmatic issue that intuitionistic methodology is so awkward that in the decades it has been available, mathematicians almost unanimously have chosen not to use it. An objection of principle against intuitionism, as with ZF, is that it also makes simple things

complicated. In intuitionism, the simple Boolean operation of binary union becomes problematic, since for some x it is possible that neither $x \in a$ nor $x \notin a$, so the extension of $x \in a \vee x \notin a$ could be less than the universe. Both resolutions of Russell's paradox fail to take advantage of the simplicity of Boolean algebra and the propositional calculus (predicate calculus without quantified formulas). Both the general theory of Boolean algebras and the propositional calculus are effectively decidable theories.

This work essentially establishes a semantics for a language which contains the words *and*, *or* and *not*, and whose rules of grammar are the rules of propositional calculus. There is scarcely any room for question about whether these rules are right or wrong. These words have standard interpretations, hence their designation as "logical constants." The propositional rules essentially define these words, so they cannot be wrong. They express the choice of using this language. Since model theory requires a restricted domain for any model of a logical language, the notion of existence is not a logical constant in the same sense that *and*, *or* and *not* are. So the thesis of this work implies that, when one uses the propositional language to discuss the more complex notion of quantification, any conflict between apparent rules and any other difficulties which arise will be due to misconceptions and incorrect assumptions about the more subtle notion of existence or existential quantification.

To continue this comparison with other set theories which imply a universal set, consider Quine's theory NF. It is by far the most respected candidate for a foundational theory with a universal set, though its consistency is not known. Quine derived NF from Russell's theory of simple types by a device which Quine himself acknowledged as a formal device with no philosophical justification [4]. Type theory resolves the paradox by assignment of a rank index to every variable. Variables of rank $n+1$ can only contain objects of rank n. So nothing contains itself. This theory

implies non-extensionality, since every rank has a different empty set. Moreover, in practice, it is prohibitively awkward to carry the rank index on every variable.

To simplify this situation Quine defined a formula to be *stratifiable* if and only if its variables can be assigned natural number indices so that variables on opposite side of equality have the same index, while a variable on the right of $\in$ has index one greater than the variable on the left. Different occurrences of the same variable must have the same index. So $x \in x$ and $x \notin x$ are unstratifiable, as are $x \in y \wedge y \in x$ and $x \in y \wedge x \neq y$. Quine's New Foundations consists of the axiom of extensionality with the axiom scheme of comprehension for stratifiable formulas. In NF there is a single empty set and a single universal set, defined by $x = x$. Once one observes that a formula is stratifiable, it is no longer necessary to retain the indices on variables.

One indication of the strength of strictly quantifier-free comprehension is a comparison with NF. The only stratifiable atomic formulas which are not strictly quantifier-free are $x = x$. The other stratifiable atomic formulas are $x = y$ and $x \in y$, both of which are strictly quantifier-free. The formula $x = x$ is logically equivalent to $x \in \emptyset \vee x \notin \emptyset$, so the strictly quantifier-free comprehension scheme with parameters gives the extension of $x = x$. Thus a model of strictly quantifier-free comprehension, which is closed under Boolean operations, will satisfy comprehension axioms for all stratifiable atomic formulas. The model also satisfies closure under cartesian products and certain variable permutations which Gödel used in the construction of L. Consequently, the model will satisfy comprehension axioms for all stratifiable quantifier-free formulas. In fact, the notion of strictly quantifier-free atomic formula could have included $x = x$ and excluded only $x \in x$, but the uniform treatment of all atomic formulas emphasizes that there is an intuitive constructive principle which the theory strictly quantifier-free comprehension follows.

Consistency of NF is a long-standing open problem, in spite of considerable at-

tention. Jensen [5] proved the consistency of Quine's comprehension scheme without extensionality. That theory is NFU, New Foundations with Urelements. Urelements are objects other than sets. So urelements contain no members, but they are not equal to the empty set. If one wishes to treat natural numbers as urelements, then NFU may be useful. However, it was not intended that the comprehension scheme force the conclusion that urelements exist.

Resolution of the consistency problem for NF is not so simple as just to say that extensionality is insignificant. Intuitively, extensionality is a property of sets. It is certainly convenient to work in a universe where everything is a set. One might accept a theory like NFU as true in the real world, and then separate the set structure as a context for mathematical construction, but then to use NF in that extensional structure of sets, a consistency proof for NF would still be necessary.

Specker [6] has proven $NF \vdash\sim AC$, though the theory of simple types is known to be consistent with the axiom of choice [4]. Yet NF is closely related to type theory in more than just its historical derivation. The theory NF is equiconsistent with type theory plus an axiom which says essentially that all different types are elementarily equivalent [7]. (This dismisses models where objects of one type have some intrinsic distinction from others.) Thus Specker's result stands as the main source of doubt about the consistency of NF. It indicates again that there is something poorly understood in the modification Quine introduced to type theory.

More fundamentally, the axiom of choice is generally regarded as a true statement about sets. In this intuition, of course, one sets aside the possible development of some computational interpretations of set theory in which the axiom of choice is not true. But such considerations are speculative. In the natural interpretations of set theory, even if NF does not contradict itself, it still contradicts the truth.

There are serious problems in application of NF. Inductive constructions are

given by comprehension axioms which say there exists a sequence whose successive elements are derived in some specified way. Quine's theory does not generally support inductive proofs which involve unstratified formulas. In some cases an alternate proof exists which does not involve unstratified comprehension. However, NF does not prove that the set of natural numbers preceding n has cardinality n. The reason is that the formula $\{m | m < n\} \notin n$ is not stratified [4].

Some arguments, which have been accepted as general, only apply to small sets in NF. For example, if NF is consistent, it does not prove Cantor's theorem that $|x| < |\mathcal{P}(x)|$ for the universal set. The usual simple proof of Cantor's theorem involves an unstratifiable formula. The formula $x = y$ is stratified, and it defines the identity relation, which is a one-to-one correspondence between the universe and its power set, which is the universe itself. Limitation of accepted constructions to small sets is a re-introduction of the limitation of size doctrine, a significant flaw in a proposed alternative to ZF.

Some theorems, such as Cantor's, are actually refuted in NF. Others are just independent of NF and can be re-instated by additional axioms. This is not an attractive solution, since much of the appeal of NF began with its return to a single principle of set formation. By itself, that contrasts favorably with the two axiom schema in ZF plus axioms of pair, union and power set formation.

Unstratifiable formulas are not uncommon in mathematics [8]. They are not so easy to recognize as, say, non-effective definitions, which Church's thesis allows mathematicians to distingiush by intuition. On the other hand, stratifiability is defined for a formula in the membership language. Statements in measure theory require a great deal of expansion in order to apply that definition. A foundational theory could reasonably require mathematicians to recognize when their constructions are not effective and to take some limited precautions in such situations. Mathemati-

cians might accept such requirements with a reasonably limited frequency of errors. Indeed, effectiveness has long been a matter of substantial concern anyway. On the other hand, reliable recognition of unstratifiable formulas would seem to sacrifice much of the ease of application of the membership language.

The basic question is whether NF is a package that non-foundational mathematicians will buy. Even if it's consistency were proven, there remain problems of awkwardness in application, weakness and outright incorrectness. In fact, as with intuitionism, the bottom line is that in the decades NF has been available, analysts and algebraists have not shown any substantial interest in its use.

Another theory to consider here is quantified Boolean algebras. They are also called cylindric algebras, since quantification corresponds to the geometrical generation of a cylinder from a set of points. Halmos studies them under the name polyadic algebras. Chapter 2 contains a simple model of the language of set theory in which the elements form a Boolean algebra. The construction is similar to a term model, as in Henkin's proof of the completeness theorem, or as in many constructions of algebras with generators and relations. This construction does little other than what the axioms of Boolean algebras say one must do to satisfy them. One observes at each step in the construction that the axioms do not require anything impossible. Can a similar model of the theory of quantified Boolean algebras resolve Russell's paradox?

Consider what the axioms of quantified Boolean algebras require. A quantified Boolean algebra is a structure $\mathcal{A} = \langle A, +, \cdot, -, \mathbf{0}, \mathbf{1}, \mathsf{c}_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$. The reduced structure $\langle A, +, \cdot, -, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra. In addition, for all κ and λ less than some α, which is the dimension of the algebra, there are cylindrification operations c_κ and diagonalization elements $d_{\kappa\lambda}$. One may think of formulas with variables x_κ. The cylindrifications correspond to quantifiers $\exists x_\kappa$ on the different variables x_κ, and

the diagonalization elements correspond to formulas $x_\kappa = x_\lambda$ or $x_\kappa = x_\kappa$. They satisfy these axioms.

(1-2)
$$c_\kappa(0) = 0 \qquad x \leq c_\kappa(x) \qquad d_{\kappa\kappa} = 1$$
$$c_\kappa(x \cdot c_\kappa(y)) = c_\kappa(x)c_\kappa(y) \qquad c_\kappa(c_\lambda(x)) = c_\lambda(c_\kappa(x))$$
$$\text{If } \kappa \neq \lambda, \text{ then } d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})$$
$$\text{If } \kappa \neq \lambda, \text{ then } c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$$

How might one find a membership relation over such a structure? The Boolean algebra has an order which has inclusion as a natural interpretation. A singleton function $S(\cdot)$ over an atomic Boolean algebra C gives a natural membership relation. If C is an atomic Boolean algebra and S is any one-to-one map of C to its atoms, then $xEy \leftrightarrow S(x) \leq y$ defines a reasonable membership relation. In any comprehension scheme of the major set theories, the formula $x = p$ with parameter p yields a set, the singleton of the arbitrary parameter p. So set structures are always assumed to be closed under singletons. As shown in Chapter 2, such a function $S(x)$ satisfies the definition for the singleton of x, and the meet and join operations of the Boolean algebra satisfy the definitions of the set theoretic intersection and union. There is no inconsistency in this. There are cylindric algebras of cardinality κ with κ atoms. We can define a singleton function and membership relation over such algebras.

How does this theory avoid inconsistency due to Russell's paradox? There is no membership relation in the algebra. Assume $\forall x[xEe \leftrightarrow \exists y, z[x = (y, z) \wedge yEz]]$, which describes an element e which is (a code for) the membership relation. Similarly, assume $\forall x[xEd_{12} \leftrightarrow \exists y, z[x = (y, z) \wedge y = z]]$, which says that the element d_{12} is (a code for) the identity relation. Then it seems natural to define r to be $c_1(d_{12} \cdot \sim e)$.

One axiom of the theory is $x \leq c_\kappa(x)$. So Russell's set would contain all the ordered pairs which belong to $d_{12\cdot} \sim e$. Thus cylindrification need not exactly correspond to binding a variable by quantification. In fact, a simple type of model for the theory of cylindric algebras is a small set of relations over a set D where, for some α, every relation in the algebra is a subset of D^α. Then cylindrification terms are not interpreted in a way which gives the extension of $x \not\equiv x$.

Another line of research in set theory with a universal set follows work of Church [9]. He argued that there is motivation to pursue such theories, proposed a list of axioms which seem reasonable for such theories and constructed a model which satisfies some of them. In particular, it satisfies axioms of extensionality and choice and schema of separation and replacement for well-founded sets (hence, also, pair formation). The model is closed under binary union and complement, and contains the set of its finite ordinals. For well-founded sets, the structure is also closed under power sets and an expanded notion of cardinal class formation, where two sets have a one-to-one correspondence in which corresponding elements have the same size. The model does not have the membership relation as an element, and does not satisfy comprehension axioms for non-well-founded sets. Church observed that his models of set theory with a universal set could also be thought of as Boolean algebras with a singleton function.

Emerson Mitchell continued this direction [10]. He constructed an extensional model which also satisfies the replacement scheme for well-founded sets and is closed under complements and power set formation. The substructure of well-founded sets in his model is isomorphic to V. Unfortunately, the structure is not closed under binary intersection and union. This surely was desired, but closure under intersection produces many small sets. When these are complex, it is not possible in the construction to recognize which ones will eventually be empty. So such constructions

tend to admit empty non-zero elements. To retain extensionality, Mitchell sacrificed some basic Boolean operations in this construction.

Dana Scott has studied structures where a Boolean algebra is endowed with a membership relation via a singleton function. He concentrated on inverse limits of finite algebras to develop a notion of profinite Boolean algebra. In those structures the natural number structure is non-standard, so they are not good candidates for a mathematical universe.

One may investigate the self-descriptive aspects of category theory in an attempt to develop set theory with a universal set. The structure of categories is a category. The operations assumed in category theory are among those used in set theory. In the category of sets, as we found in the study of quantified Boolean algebras, if we add the assumptions necessary for the traditional strength of set theory, then the outright inconsistency due to Russell's paradoxical set arises again.

A recent development in the field is positive set theory, or topological set theory. The theory describes a lattice structure of sets closed under union and intersection. Since the order induces a topology, one can meaningfully approximate the complement of a set. Again, this work takes the Boolean operations as most fundamental in logic, and takes their effectiveness to be part of their nature as rules of grammar for a simple language. The necessity of approximation to represent complements seems far from the intuition suggested by the ideal set theory.

Oberschelp [11] has developed some theories which combine various features of ZF and NF. These theories are consistent, but they are quite weak in their treatment of large classes, as are most theories. His theory $ZF+M$-Comprehension has a superficial similarity to strictly quantifier-free comprehension. In his M-Comprehension scheme, he describes an assignment of an index 0 to element variables (those which occur on the left side of $\in$) and 1 to class variables (those which occur on the right

side of $\in$). A variable which receives both indices in different occurrences is restricted to small classes. This again uses the limitation of size doctrine to rescue the theory when the ideas intended to resolve the paradoxes fail to do so.

There is a program of reverse mathematics, initiated by Friedman [12] and by Simpson, Friedman and Smith [13], which develops substantial parts of algebra and analysis in second order arithmetic, where the sets form a Boolean algebra. That program uses theories with weak comprehension schema, often weaker than Π^1_1-CA or even Σ^0_1-CA, so not every relation is assumed to have a range. Why not extend the program of reverse mathematics in its arithmetic setting? One objection is that it is very awkward to encode all constructions in arithmetic. Second, the validity of theorems in some cases depends on the use of encodings different from those originally chosen. Such results may be seen as theorems about the codes, not about the mathematical universe. Third, the restriction to second order structures rules out the use of second order parameters in comprehension axioms. One consequence of this is that, while the axiom scheme Σ^0_1-CA implies Σ^0_n-CA for every $n \in \omega$, at the same time Π^1_1-CA does not imply Π^1_n-CA for $n > 1$. In set theory with a genuine universal set, the confrontation of Russell's paradox forces a conceptualization where the ZF rank of sets is irrevelant. Finally, even in the weakest theories of reverse mathematics, comparable to effective analysis, it is provable that the set of reals is uncountable. These theories prohibit a refinement of the notion of uncountability within the theory itself.

The construction of a model of strictly quantifier-free comprehension will develop as follows. Chapter 2 introduces most of the basic ideas of the main constructions, which are atomic Boolean algebras with singleton functions.

DEFN 1.3. Define an S-*algebra* $\langle C, \wedge, \vee, \sim, \mathrm{S} \rangle$ to be any atomic Boolean algebra

with a function $S : C \xrightarrow{1-1} C$, where the values of S are atoms of C. The *associated membership relation* for any $y, z \in C$ is defined by yEz iff $S(y) \leq z$. Then the maximum element of the Boolean algebra contains all elements of the algebra, and we denote this universal element by u.

It is useful to distinguish the membership structure $\langle C, E \rangle$ of an S-algebra because sometimes it is necessary to refer to the associated membership relation before we have completed closure of a structure under $S = \bigcup\{S_m\}_{m \in \mathbf{N}}$. The constructions which follow use finite Boolean subalgebras $B_m < {}^{\mathbf{N}}2$ with a function S_m which maps each element of B_m to an atom of ${}^{\mathbf{N}}2$. Some of those atoms are outside of B_m. Still $xE_m y \leftrightarrow S_m(x) \leq y$ defines a membership relation between any $x \in B_m$ and any $y \in {}^{\mathbf{N}}2$. Even though B_m is not closed under S_m, the membership structure $\langle B_m, E_m \rangle$ does satisfy closure under Boolean operations. Upon iteration to close under singletons, that membership relation is the associated membership relation for the singleton function S so defined.

Section 2.1 contains the basic definitions and some basic constructions and results about these structures. There are basic Boolean algebras with singleton functions, and one contains an element e which codes the membership relation for the entire structure. Section 2.2 contains a more complex but finitely generated S-algebra which introduces most of the ideas, structural features and terminology used in the model of strictly quantifier-free comprehemsion with internal countabiltiy, which is in Section 4.2. Section 2.3 contains two theorems. First, the class of singleton algebras is closed under direct limits. Second, if a structure is closed under certain familiar "Gödel-like" operations, then it satisfies strictly quantifier-free comprehension.

Chapter 3 contains a theorem that any S-algebra closed under these Gödel-

like operations contains an extensional S-subalgebra which preserves that closure property.

Chapter 4 begins with a preliminary discussion in Section 4.1 of the main construction in Section 4.2 of an internally countable S-algebra closed under Gödel-like operations. That structure is a direct limit of finitely generated S-algebras. Section 4.3 is the verification that this system constructed is indeed a direct limit.

Chapter 5 summarizes the main features of the construction and its method. There is also a brief description of a program which promises to extend this methodology to a resolution of the Russell and Cantor paradoxes as discussed in the presence of an axiom of replacement strong enough to support substantial algebra and analysis.

CHAPTER 2
BASIC CONSTRUCTIONS

In this chapter, we will use various relatively simple S-algebras constructions to become familiar with a method of construction which we repeat in Chapter 4, essentially as one step in a direct limit system of structures. The main example of this chapter, in section 2, will contain the distinguished features listed in (2-1) below, each of which will satisfy the corresponding property in (2-2). The incorporation of these features will use almost all the essential ideas of the complicated direct limit in Chapter 4, which has infinitely many similar features. We will finish this chapter with section 3 which contains some general results about S-algebras.

DEFN 2.1. In a membership structure $\langle C, E\rangle$, we will define the following *distinguished elements.*

(2-1)

u is the universal set

e is the membership relation E

u^0 is the set of all elements which are not ordered pairs

ω is the set of all natural numbers for the structure

f is a 1-1 function which maps u into ω

$\mathcal{P}(\omega)$ is the power set of ω

$\mathcal{P}(e)$ is the power set of e

We will avoid the notation u^1 since in Chapter 4 we will use u^2 and u^3 to denote elements which contain, respectively, all ordered pairs and all ordered triples of the structure.

We use (y,z) as an abbreviation for $S((y)) \vee S(S(y) \vee S(z))$, the usual code for an ordered pair as coded in the model. So the structure is closed under ordered pairs, and the ordered pair function will be one-to-one because S will be one-to-one. We will use $\langle y,z \rangle$ to denote an actual ordered pair in V, for example, the model $\langle C, E \rangle$ refered to in Defn 2.1. We will denote the natural numbers of the real world by $\mathbf{N}$ to avoid confusion with the element ω of the model. With the membership relation induced by $yEz \leftrightarrow S(y) \leq z$, we can then describe the desired properties of the distinguished features as follows.

(2-2)

$$xEu \qquad \text{for all } x$$

$$xEe \leftrightarrow \exists y,z \in B[x = (y,z) \wedge yEz]$$

$$xEi \leftrightarrow \exists y,z \in B[x = (y,z) \wedge y = z]$$

$$xEu^0 \leftrightarrow \nexists y,z \in B[x = (y,z)]$$

$$\langle \{x \in B | xE\omega\}, E \} \rangle \cong \langle \mathbf{N}, \in \rangle$$

$$\forall x,y,z[(x,y)Ef \wedge (x,z)Ef \rightarrow y = z$$

$$\forall x \exists y E\omega\,[(x,y)Ef]$$

$$\forall x,y[(x,y)Ef \rightarrow \forall z[(x,z)Ef \leftrightarrow z = y]]$$

$$xE\mathcal{P}(\omega) \leftrightarrow x \leq \omega$$

$$xE\mathcal{P}(e) \leftrightarrow x \leq e$$

2.1. Basic Constructions

These introductory examples will be Boolean subalgebras of a familiar atomic Boolean algebra, $\langle {}^{\mathbf{N}}2, \wedge, \vee, \sim \rangle$, the set of functions from $\mathbf{N}$ into $\{0,1\}$. Our Boolean operations are inherited in that context so long as our singletons are atoms of ${}^{\mathbf{N}}2$.

Note that atoms of a subalgebra C may not be atoms of ${}^{\mathsf{N}}2$. To minimize confusion, we will generally use atoms of ${}^{\mathsf{N}}2$ and clearly note the exceptions. If we have a subalgebra $C \leq {}^{\mathsf{N}}2$ with a singleton function $\mathrm{S} : C \xrightarrow{1-1} C$ such that for every $x \in C$ the singleton $\mathrm{S}(x)$ is an atom of ${}^{\mathsf{N}}2$, then for all $x, y \in C$ we define a membership relation $xEy \leftrightarrow \mathrm{S}(x) \leq y$. In that case, since $\mathrm{S}(x)$ is an atom of ${}^{\mathsf{N}}2$, we have $\mathrm{S}(x) \leq y \wedge z$ iff $\mathrm{S}(x) \leq y$ and $\mathrm{S}(x) \leq z$. So $y \wedge z$ is $y \cap z$ relative to $\langle C, E\rangle$. Closure under union and complement are similar. Thus $\langle C, E\rangle$ satisfies closure under complement and binary union and intersection, and the Boolean operations of ${}^{\mathsf{N}}2$ do play their intended roles. Moreover, since S is one-to-one, for any $x, y \in C$ the structure $\langle C, E\rangle$ satisfies $y E \mathrm{S}(x) \leftrightarrow y = x$. Thus, our terminology of singleton function is appropriate, and we formalize the following definition.

DEFN 2.2. For any S-algebra $\langle C, \wedge, \vee, \sim, \mathrm{S}\rangle$, any set X included in C and any element $x \in C$, the smallest S-algebra in C which contains X and x is the S-*subalgebra generated by* X *and* x; denote it by $\mathrm{SA}(X, x)$. We use $\mathrm{BA}(X, x)$ to denote the *Boolean subalgebra generated by* X *and* x. When we refer to a subalgebra generated by X and x without specifying whether it is an S-subalgebra or a Boolean subalgebra, it will be the Boolean subalgebra generated by X and x. If X is a Boolean algebra, then we use $X(x)$ for $\mathrm{BA}(X, x)$. Since the associated membership relation for any $y, z \in C$ is defined by yEz iff $\mathrm{S}(y) \leq z$, the maximum element of the Boolean algebra contains all elements of the algebra, and we denote this universal element by u.

EXAMPLE 2.3. We construct an S-algebra, one which happens to be the minimum S-algebra. We will choose a Boolean subalgebra of $\langle {}^{\mathsf{N}}2, \wedge, \vee, \sim\rangle$ and augment the structure with a singleton function. To avoid confusion with elements of ω, we will use $\mathbf{0}$ and $\mathbf{1}$ to denote the identities of our Boolean algebras. Thus u will be $\mathbf{1}$. Let $A < {}^{\mathsf{N}}2$ be the Boolean subalgebra generated by the atoms $\chi_{\{j\}}$ for all $j \in {}^{\mathsf{N}}2$,

where $\chi_{\{j\}}$ is the characteristic function of the singleton of j. Enumerate A without repetitions as $\{x_j\}_{j\in\omega}$. Define $\mathrm{ind}(x_j) = j$, so $x_{\mathrm{ind}(x)} = x$, and ind maps A one-to-one onto ω. Define $\mathrm{S}(x) = \chi_{\{\mathrm{ind}(x)\}}$.

Now $\langle A, \wedge, \vee, \sim, \mathrm{S}\rangle$ is an S-algebra since S is one-to-one and its values are atoms. This structure is closed under formation of finite and co-finite sets, including $\emptyset = \mathbf{0}$ and $u = \mathbf{1}$. It contains only the finite and co-finite sets. It includes a copy of V_ω and it is extensional. Still it is uninteresting. As we make the algebra richer and more useful, of course, it is more difficult to achieve extensionality.

The definition of S-algebras does not require that all atoms of $^{\mathsf{N}}2$ will be used by S. That would trivialize the problem of extensionality, so it is not generally possible. The main theorem of Chapter 3 gives an extraction of an extensional substructure from an S-algebra which preserves the features of concern in this work, and it is generally very much easier to separate the extensionality problem from other restrictions on construction of an S-algebra. The definition also does not require that the singleton function S map onto all the atoms of $^{\mathsf{N}}2$ which are in the subalgebra C. However that will be the case in our constructions, since the following problem can arise otherwise.

Suppose we define $B = \cup\{B_m\}_{m\in\mathsf{N}} < {}^{\mathsf{N}}2$ by induction on m. We will formally define $B_{m+1}\backslash B_m$ to be the m^{th} rank of B. If an object in B_m has support in $^{\mathsf{N}}2$ of cardinality one or two while the object has no members at a stage B_m in our construction, then it is not an ordered pair in B_m, and S_m will place it in u^0, hence in the complement of any relation such as e or $\sim e\backslash u^0$. If the object turns out to be an ordered pair at a later stage, we could not then treat it as an ordered pair and maintain a compatible system of singleton functions. This is why we generally construct each algebra B as $\bigcup\{B_m\}_{m\in\mathsf{N}}$ by an induction on m, and we start with

a sparse subalgebra of $^{\aleph}2$, i.e. a algebra whose elements differ by infinitely many atoms. In such constructions, there are no ordered pairs initially, since they must have support of size one or two. As we iterate to close under singletons, we produce ordered pairs only via singletons of previously constructed elements. So when we produce an ordered pair in the construction we are in a position to recognize the ordered pair immediately and define its singleton appropriately.

The next proposition verifies that there is a minimum S-algebra, and it is an introduction to the important notion of ranking relative to a generator algebra. Since structures with a universal set involve self-description, especially those with a universal membership relation, we need to be careful in our treatment of this characteristic. The way we accomplish this is by induction on ranks relative to a generator algebra. Our constructions use the fact that we begin with atomic subalgebras in $^{\aleph}2$ whose minimal non-zero elements have infinite support, so sup's of finitely many of the singletons we define are never elements of the generator algebras.

DEFN 2.4. Let C be an S-algebra with a Boolean subalgebra $Q < C$. Suppose that for all $x_1, \ldots, x_k \in C$, we have $\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k) \notin Q$. The *ranks of C with respect to Q* are $R_0 = Q$ and $R_{m+1} = R_m(\mathrm{S}[R_m])\backslash R_m$, which is the set of new elements of the Boolean extension of R_m by the set of singletons $\mathrm{S}[R_m]$. So the m^{th} rank is the set of new elements introduced at stage m. The *rank* $\rho(x)$ is the least m such that $x \in R_m$. We use C_m to denote $R_0 \cup \cdots \cup R_m$ and, for uniformity, C_ω denotes C.

PROPN 2.5. *If C is an* S*-algebra then it has an* S*-subalgebra M which is included in each of its other* S*-subalgebras. If C and C' are* S*-algebras, then their minimal* S*-subalgebras M and M' are isomorphic.*

PROOF: Among the S-subalgebras of C is C itself, and the intersection M of all

S-subalgebras of C is closed under $\vee$, $\wedge$, $\sim$ and S, i.e. it is an S-subalgebra. It is included in all S-subalgebras of C, so it is the minimum S-subalgebra in C.

We will represent M as the union of Boolean algebras M_m, which we can naturally define as its ranks with respect to the generating Boolean subalgebra $\{\mathbf{0}, \mathbf{1}\}$.

Let S be the singleton function of C, and let C' be another S-algebra with singleton function S' and minimum S-subalgebra M'. We define $M_0 = \{\mathbf{0}, \mathbf{1}\}$ and define M_{m+1} is the Boolean algebra $M_m(\mathrm{S}(x) | x \in M_m)$. Since S is one-to-one, we know $\mathbf{0}$, $\mathrm{S}(\mathbf{0})$, $\mathrm{S}(\mathrm{S}(\mathbf{0}))$, etc. are distinct, and the subalgebra M is infinite. Now consider the set of elements which are either finite and equal to $\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k)$ for some $x_1, \ldots, x_k \in M_m$, or are co-finite and equal to $\sim (\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k))$, for some $x_1, \ldots, x_k \in M_m$. Elements of this set are uniquely erpresentable by one of these forms. The set is closed under $\wedge$, $\vee$, S and $\sim$. Thus, we have simple unique normal forms in M. So we can define $\psi_0(\mathbf{0}) = \mathbf{0}'$ and $\psi_0(\mathbf{1}) = \mathbf{1}'$. Let

$$\psi_{m+1}(\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k)) = \mathrm{S}'(\psi_m(x_1)) \vee \cdots \vee \mathrm{S}'(\psi_m(x_k)) \tag{2-3}$$

and

$$\psi_{m+1}(\sim (\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k))) = \sim (\mathrm{S}'(\psi_m(x_1)) \vee \cdots \vee \mathrm{S}'(\psi_m(x_k))) \tag{2-4}$$

The total function ψ is $\cup\{\psi_m\}_{m \in \omega}$.

Since S and S' are one-to-one, the map ψ is well-defined and one-to-one. Since M' is minimal, the map ψ is onto M'. ∎

Note some important features in this construction. We start with a Boolean subalgebra Q of ${}^{\mathbf{N}}2$ whose non-zero elements all have infinite support. This guarantees

that neither the non-zero elements of Q, nor their Boolean combinations with the singletons we generate, will become ordered pairs of elements generated later, since an ordered pair is a sup of just one or two atoms of ${}^{\mathbf{N}}2$.

DEFN 2.6. For any $x \in {}^{\mathbf{N}}2$ the *support* of x is

$$\operatorname{supp}(x) = \{j \in \omega | \, x(j) = 1\} \tag{2-5}$$

DEFN 2.7. A subalgebra $Q < {}^{\mathbf{N}}2$ is *sparse* if its nonzero elements have infinite support.

So, if Q is sparse then we know that elements of Q will not be ordered pairs of other elements of Q or of any objects constructed at later stages. Of course the singletons we define for elements of this sparse Q are atoms of ${}^{\mathbf{N}}2$, so they are not in Q. They generate an expanded algebra over Q, for which we need to define more singletons. As we iterate this procedure, the only way a singleton arises is if its element has been previously defined. Thus, we do not encounter $x = \mathrm{S}(x)$ or related situations such as $x = (x, x)$.

We proceed with an example to demonstrate that an internal code for the entire membership relation can actually be a simple matter, when we construct an S-algebra by ranks from a generating algebra. We will define a Boolean algebra $B = \bigcup\{B_m\}_{m\in\omega}$ with a one-to-one singleton function S which maps B to its atoms. The algebra B will contain finite and co-finite sets, and elements e and $\sim e$. The element e will enumerate all elements of the structure as cross-sections, i.e. it satisfies $\forall x[xEe \leftrightarrow \exists y, z[x = (y, z) \wedge yEz]]$. The S-algebra B will also be extensional.

EXAMPLE 2.8. In $\langle {}^{\mathbf{N}}2, \wedge, \vee, \sim \rangle$ let $e = \chi_{2\mathbf{N}}$, the characteristic function of the even numbers, and let $B_0 = \{0, 1, e, \sim e\}$. As in all constructions in this work, we choose

a countable environment A which will contain all objects we might need as elements of our S-algebra. This is because we only have countably many singletons in $\aleph_2$ as potential placements of $S(x)$. The environment A may have unnecessary elements as long as there are only countably many of them. Take A to be the Boolean algebra generated by B_0 with all the atoms of $\aleph_2$. Enumerate A without repetitions as $A = \{x_j\}_{j\in\omega}$. Define ind : $A \xrightarrow[\text{onto}]{1-1} \omega$ by $\text{ind}(x_j) = j$, so $x_{\text{ind}(x)} = x$.

In the first two steps of the inductive definition of S, we use the fact that elements of infinite support are not ordered pairs. Similarly their singletons and doubletons are not ordered pairs either. For all $x \in B_0$ let $S_0(x)$ be the $\text{ind}(x)^{th}$ atom under $\sim e$. For all $n \in \omega$ define B_{m+1} to be the Boolean algebra generated by B_m and its singletons, $S_m[B_m]$). For all $x \in B_1$ let $S_1(x)$ be the $\text{ind}(x)^{th}$ atom under $\sim e$, since again no $x \in B_0$ is an ordered pair.

Each stage B_{m+2} will contain ordered pairs, and some should become members of e. For all $m \in \omega$ and all $y, z \in B_m$ define $(y,z)_m = S_{m+1}(S_m(y)) \vee S_{m+1}(S_m(y) \vee S_m(z))$. Note that we use the index of a function to indicate its domain. So if $(y,z)_m$ is defined, then y and z are in B_m. Conveniently, the singleton $S_m(y)$ is in B_{m+1}, and we can observe whether $S_m(y) \leq z$. So for all $x \in B_{m+2}$ we can define S_{m+2} by

(2-6) $S_{m+2}(x) = \text{the } \text{ind}(x)^{th} \text{ atom under } e, \quad \text{if } \exists y, z \in B_{m+1}[x = (y,z)]$

$S_{m+2}(x) = \text{the } \text{ind}(x)^{th} \text{ atom under } \sim e, \text{if } \nexists y, z \in B_{m+1}[x = (y,z)].$

Finally, let $B = \bigcup\{B_m\}_{m\in\omega}$

The requirement that the singleton function be one-to-one is easily satisfied. Each (infinitely supported) atom relative to B_0 allows us to place any number of distinct singletons under it, which we do by use of the one-to-one index function.

Note that since S is one-to-one, we have infinitely many elements $y = S^n(0)$. So we have infinitely many elements $(y, 1)Ee$, and we have infinitely many $(y, 0)E \sim e$. It follows that B is extensional.

In one sense, the element e is the most complicated in the universe. In another sense, its construction requires no new information beyond what is required for the sets it enumerates. This is similar to the construction in recursion theory of an effective simulator for all effectively computable functions. To simulate the simulator, when it applies to a function paired with an argument, we just simulate that function on that argument. In that sense, the simulator does not act as a complicated function. It acts as a non-operation.

The simplicity of e in our constructions is due to the following two facts. When we define $S_{m+2}(y, z)_m$ and we need to decide whether to place it under e or not, we have already defined $S_m(y)$. In addition, we are in an environment with an order derived from the Boolean algebra ${}^{N}2$. So to decide the placement of $S_{m+2}(y, z)_m$, we just need to observe whether $S_m(y) < e$ or not. In a construction in Chapter 4, we use the fact that, for $x \in B_m$ and any element p of ${}^{N}2$, we can compare $S_m(x)$ with p, even if $p \notin B_m$. This is an advantage associated specifically with the Weiner-Kuratowski style codes of ordered pairs.

To construct an algebra with e, i and u^0, we could easily make an extensional model by exploiting some simple observations. The members e and u^0 will have no common elements. Similarly i and u^0 have no common elements. The pairs e and i, e and $\sim i$, $\sim e$ and i, and $\sim e$ and $\sim i$ each have infinitely many common elements. Rather than rely on such observations, we choose a more generic construction which is typical of our more complicated models.

In complicated constructions we will not want to burden ourselves with anticipation of how many, if any, elements an object will have in the end. In the next

chapter, we will present a general method which extracts a rich, "comprehensive" extensional substructure from any S-algebra. While the comprehensive substructure will not be an elementary substructure, which would preserve non-extensionality, it will be a rich enough substructure to preserve the features listed in (2-1). So we will allow non-extensionalities to arise in our construction.

To finish consideration of this model, we note two aspects of the passage to an extensional substructure. First, we refer in this chapter to "power sets" of ω and e. In a non-extensional membership structure associated with a Boolean algebra, we will find for some x that $\forall z[zEx \rightarrow zE\omega]$ without $z \leq \omega$, since $\omega \backslash z$ may be non-zero and empty. We can omit such z from $\mathcal{P}(\omega)$ since that situation will disappear in the extensional substructure.

A second consequence of non-extensionality relates to undesired elements which may qualify as natural numbers of the structure if we define the natural numbers by their extensions. We select a natural representative of each $n \in \omega$ based on the formulas $0' = 0$ and $(j+1)' = j' \vee \mathrm{S}_m(j')$ for $j = 0, \ldots, m-1$. These elements are distinct because ind and hence S_m are one-to-one. These elements will have a structure isomorphic to $\langle \omega, \in \rangle$, and they will be present in any S-subalgebra. The undesired elements will disappear in the extensional substructure. Thus, the element denoted ω will be isomorphic to the real natural numbers and, in the end, it will contain all the natural numbers of the extensional model.

2.2. The Main Construction

We begin our main construction of this chapter. It has several aspects. One is to pick some elements of ${}^{\mathbf{N}}2$ which dominate an adequate number of atoms, and whose Boolean combinations also dominate an adequate number of atoms. Then we choose a countable environment to facilitate the last steps, which are to define the

initial and successive stages of an inductive definition of a singleton function. Finally, of course, we must verify the properties we claim for the distionguished features of the algebra. Throughout this section we will introduce terminology which is used in more complex constructions, though it is perhaps not necessary for this construction itself.

The complex parts of the construction are those in which, for every element x of the structure, we choose an appropriate atom of $^{\aleph}2$ to be its singleton, $S(x)$. By "appropriate" we mean that the distinguished features play their intended roles. Recall that this means B must satisfy the following.

(2-2)
$$xEu \quad \text{for any } x$$
$$xEe \leftrightarrow \exists y, z \in B[x = (y, z) \wedge yEz]$$
$$xEi \leftrightarrow \exists y, z \in B[x = (y, z) \wedge y = z]$$
$$xEu^0 \leftrightarrow \not\exists y, z \in B[x = (y, z)]$$
$$\langle \{x \in B | xE\omega\}, E\} \rangle \cong \langle \omega, \in \rangle$$
$$\forall x \exists y E\omega[(x, y)Ef] \text{ and } \forall x, y[(x, y)Ef \rightarrow \forall z[(x, z)Ef \leftrightarrow z = y]]$$
$$xE\mathcal{P}(\omega) \leftrightarrow x \leq \omega$$
$$xE\mathcal{P}(e)] \leftrightarrow x \leq e$$

We have seen that correct treatment of ordered pairs is problematic. The question which arises repeatedly in these constructions is which problems in the comprehension axioms can be resolved by the nice characteristics of Weiner-Kuratowski pairs and our notion of rank relative to a generator algebra. Note, incidentally, that our purpose is to prove the consistency of some axiom systems which includes an axiom that the

ordered pair function is one-to-one. The usual codes for ordered pairs are just one choice, which happens to be convenient for the method of construction we will use.

The Generators

We can easily pick a finite number of generators all of whose non-zero Boolean combinations dominate infinitely many atoms. For example, take the characteristic functions of the first seven prime ideals in the natural numbers $\mathbb{N}$. This choice may result in non-extensionality, but that can be rectified as we show in Chapter 3.

DEFN 2.9. In ${}^{\mathbb{N}}2$ we define the *distinguished features*

$$(2\text{-}7) \qquad e = \chi_{2\mathbb{N}} \qquad i = \chi_{3\mathbb{N}} \qquad u^0 = \chi_{5\mathbb{N}} \qquad \omega = \chi_{7\mathbb{N}} \qquad f = \chi_{11\mathbb{N}}$$
$$\mathcal{P}(\omega) = \chi_{13\mathbb{N}} \qquad \mathcal{P}(e) = \chi_{17\mathbb{N}}$$

Since our structure will be countable, the Boolean combinations of these generating objects do have room for singletons of all possible elements under them.

A Countable Environment

To organize the choice procedure, and in order to define members of the distinguished feature f, we will choose a countable subalgebra $A \leq {}^{\mathbb{N}}2$. This algebra will serve as a countable environment which contains all objects we will need in this construction and also any things we are not sure about, so long as there are only countably many of them.

DEFN 2.10. Let T be the set of distinguished features e, i, u^0, ω, f, $\mathcal{P}(\omega)$ and $\mathcal{P}(e)$. Let Q be the finite Boolean algebra generated by T. Let A be the countable Boolean algebra generated by Q with all the atoms $\chi_{\{j\}}$ of ${}^{\mathbb{N}}2$. Enumerate A without repetitions as $A = \{x_j\}_{j\in\omega}$ and define $\mathrm{ind} : \xrightarrow[\text{onto}]{1-1} \omega$ by $x_{\mathrm{ind}(x)} = x$.

The Central Ideas

Conditions (2-2) place eight restrictions on our choice of the values for the singleton function. The construction proceeds in stages indexed by m as in the previous example, 2.8. For each x, for each generator g among the distinguished features, we will decide whether to place $S_m(x)$ under g or under $\sim g$. The combined restriction which results is a non-zero Boolean combination of the generators in Q. So it has infinite support, and we can distribute infinitely many singletons under it if necessary. The first three restrictions are quite simple. First, when we encounter $x = (y,z)_m$ and need to define $S_{m+2}(x)$, we need to know if yE_mz. That is easy, since we must have defined the singleton $S_m(y)$ in the construction of $(y,z)_m$; we have the order structure of ${}^{\mathbf{N}}2$, so we can indeed answer the question whether $S_m(y) \leq z$. Second, we need to decide if x needs to be an element of u^0. We have pointed out that if we have an object x in a structure B_m and x is not the ordered pair of any two objects in stage m, then it will not be the ordered pair of any objects from any stage. Thus we can choose a singleton for it which is dominated by u^0. Our definition of the singleton at later stages by the same principle will be compatible with our current choice. Third, when we have $x = (y,z)_m$, we can observe whether $y = z$ and place $S_{m+2}(x)$ under i if required.

For the element ω, first note that for all $q \in Q$ we want $S_0(q) < \omega$ iff $q = \mathbf{0}$. At stage 1, the only natural elements of ω we will have are $\mathbf{0}$ and $S_0(\mathbf{0})$. We place $S_1(S_0(\mathbf{0}))$ under ω. We define $0' = \mathbf{0}$ and $(j+1)' = j' \vee S_m(j)$. At each stage m, the natural numbers $0', \ldots, m'$ are elements of B_m and we must define a singleton for the one new natural number m'. So we will place only $S_m(m)$ under ω with its natural number predecessors. Note that these elements are not ordered pairs (only $1'$ and $2'$ even have the correct support), so the compatibility of the different singleton

functions on these elements is not problematic. Thus, in reference to j', we will suppress reference to the stage index m.

So long as the minimal non-zero elements q of our finite generator algebra $Q < {}^{\aleph}2$ have infinite support, we can place as many objects' singletons as we need to under each q. To distribute the singleton of a given $x \in B_m$, we choose the $\mathrm{ind}(x)^{th}$ atom of ${}^{\aleph}2$ under q. (We refer here to the atoms as ordered by the map ind, but other orders would serve as well.) Of course, as we generate singletons, we will generate objects of finite support. They are the natural finite sets of our structure, and we will not try to put infinitely many elements in them.

For the element f, we use the index function a second time. As we generate and recognize an ordered pair $x = (y, z)_m$, we place $\mathrm{S}_{m+2}(x)$ under f iff $z = \mathrm{ind}(y)'$. When an element y is generated in the construction, we may not yet have $\mathrm{ind}(y)'$ in the partial structure. This is not a problem. We only need to recognize then, for each ordered pair $x = (y, z)_m$ which we have constructed, that $\mathrm{S}_{m+2}(x)$ is not to be under f.

Finally, we have the power sets of two elements. One, $\mathcal{P}(\omega)$, is closely related to the real line. The other, $\mathcal{P}(e)$, is simply the power set of the most complicated element in the model. Both are treated the same. We will place $\mathrm{S}_m(x)$ under $\mathcal{P}(\omega)$ iff $x \leq \omega$, and under $\mathcal{P}(e)$ iff $x \leq e$. As noted above, there is a problem with non-extensionality, and this will disappear in an extensional substructure. For now, we have an element which contains an extensionally equivalent representative of any set x such that $\forall z[zEx \rightarrow zE\omega]$, and similarly for $\mathcal{P}(e)$.

Note two significant characteristics of unions of Boolean algebras as compared with the more common construction of a union of unstructured sets (C_n, E_n) of elements. Mitchell's construction [10] ignores the algebraic structure available in intermediate stages. The result is that the universal quantifier in the usual definition

of the subset relation requires laborious analysis. Here, we can represent the subset relation by a quantifier-free formula. Moreover, when we extract a rich extensional substructure in Chapter 3, we use the fact that every countable Boolean algebra is projective. Thus, we find significant advantages in use of an algebraic construction similar to generators with relations, rather than construction of a Henkin model for a comprehension scheme which we have become convinced is consistent.

The Initial Stages

We now begin an inductive definition of an S-algebra B with elements which satisfy (2-2). The structure B will be the union of a chain of Boolean algebras B_m, each of which has a total singleton function S_m some of whose values are *not* in B_m. The algebra B_{m+1} will be the extension of B_m by those singletons. So B_{m+1} is $B_m(\mathrm{S}_m[B_m])$. We will use subscripts to indicate the domain of each function. For all $y, z \in B_m$, we define $(y,z)_m = \mathrm{S}_{m+1}(\mathrm{S}_m(y)) \vee \mathrm{S}_{m+1}(\mathrm{S}_m(y) \vee \mathrm{S}_m(z))$. So $(y,z)_m$ will be a member of the structure B_{m+2}.

DEFN 2.11. We will refer to the elements e, i, u^0, ω, f, $\mathcal{P}(\omega)$ and $\mathcal{P}(e)$ as *special terms*. Let T denote the set of those elements. Let B_0 be the Boolean algebra generated by T. To facilitate the definition of S_m we introduce the *restriction on* $\mathrm{S}_m(x)$ *due to* τ, which we denote $\mathrm{R}^{\mathrm{S}}_m(x,\tau) \in \{\tau, \sim\tau\}$, for each $m \in \omega$, each $x \in B_m$ and each $\tau \in T$.

As described above we will decide the value of each $\mathrm{R}^{\mathrm{S}}_m(x,\tau)$. Then for each $m \in \omega$ and $x \in B_m$, we know $\bigwedge\{\mathrm{R}^{\mathrm{S}}_m(x,\tau)\}_{\tau\in T}$ has infinite support in ${}^{\mathrm{N}}2$, so we can define $\mathrm{S}_m(x)$ to be the $\mathrm{ind}(x)^{th}$ atom under $\bigwedge\{\mathrm{R}^{\mathrm{S}}_m(x,\tau)\}_{\tau\in T}$, and continue the induction on m. The order of atoms we refer to is their order in the enumeration of elements of A.

DEFN 2.12. We define S_0 and S_1 separately from S_{m+2}. Since B_0 is sparse, none of

its elements are ordered pairs. The only natural number in B_0 is $\mathbf{0}$. Our definition of subsets of ω or of e does not depend on m, since it just refers to the order relation of ${}^{\mathsf{N}}2$. Thus we define

(2-8)
$$\begin{aligned} \mathrm{R}^{\mathrm{S}}_0(x,\tau) &= \sim\tau \quad \text{for} \quad \tau = e, i, f \text{ and any } x \\ \mathrm{R}^{\mathrm{S}}_0(x,u^0) &= u^0 \quad \text{for} \quad \text{any } x \\ \mathrm{R}^{\mathrm{S}}_0(x,\omega) &= \omega \quad \leftrightarrow \quad x = \mathbf{0} \\ \mathrm{R}^{\mathrm{S}}_0(x,\mathcal{P}(\omega)) &= \mathcal{P}(\omega) \quad \leftrightarrow \quad x \leq \omega \\ \mathrm{R}^{\mathrm{S}}_0(x,\mathcal{P}(e)) &= \mathcal{P}(e) \quad \leftrightarrow \quad x \leq e \end{aligned}$$

Let $\mathrm{S}_0(x)$ be the $\mathrm{ind}(x)^{th}$ atom under $\bigwedge\{\mathrm{R}^{\mathrm{S}}_0(x,\tau)\}_{\tau\in T}$.

Now B_1 contains elements $\mathrm{S}_0(y)$ and $\mathrm{S}_0(y) \vee \mathrm{S}_0(z)$. In such cases the elements $y, z \in B_0$ are either $\mathbf{0}$ or have infinite support, so B_1 still contains no ordered pairs. The only natural number codes present are $\mathbf{0}$ and $\mathrm{S}_0(\mathbf{0})$. Thus, we define $\mathrm{S}_1(x)$ as the $\mathrm{ind}(x)^{th}$ atom under $\bigwedge\{\mathrm{R}^{\mathrm{S}}_1(x,\tau)\}_{\tau\in T}$, where

(2-9)
$$\begin{aligned} \mathrm{R}^{\mathrm{S}}_1(x,\tau) &= \sim\tau \quad \text{for} \quad \tau = e, i, f \text{ and any } x \\ \mathrm{R}^{\mathrm{S}}_1(x,u^0) &= u^0 \quad \text{for} \quad \text{any } x \\ \mathrm{R}^{\mathrm{S}}_1(x,\omega) &= \omega \quad \leftrightarrow \quad x = \mathbf{0}, \mathrm{S}_0(\mathbf{0}) \\ \mathrm{R}^{\mathrm{S}}_1(x,\mathcal{P}(\omega)) &= \mathcal{P}(\omega) \quad \leftrightarrow \quad x \leq \omega \\ \mathrm{R}^{\mathrm{S}}_1(x,\mathcal{P}(e)) &= \mathcal{P}(e) \quad \leftrightarrow \quad x \leq e \end{aligned}$$

Note that the singleton function S_1 is an extension of S_0.

The Successive Stages

The next stages B_{m+2} do contain ordered pairs.

DEFN 2.13. We define $(y,z)_m$ to be $S_{m+1}(S_m(y)) \vee S_{m+1}((S_m(y) \vee S_m(z))$. Then

(2-10) $\quad S_{m+2}(x)$ is the $\mathrm{ind}(x)^{th}$ atom under $\bigwedge\{R^S_{m+2}(x,\tau)\}_{\tau\in T}$

where

$$
\begin{aligned}
(2\text{-}11)\qquad R^S_{m+2}(x,e) = e \quad &\leftrightarrow \quad \exists y,z \in B_m[x = (y,z)_m \wedge S_m(y) \leq z] \\
R^S_{m+2}(x,i) = i \quad &\leftrightarrow \quad \exists y,z \in B_m[x = (y,z)_m \wedge y = z] \\
R^S_{m+2}(x,u^0) = u^0 \quad &\leftrightarrow \quad \not\exists y,z \in B_m[x = (y,z)_m] \\
R^S_{m+2}(x,\omega) = \omega \quad &\leftrightarrow \quad x = \mathbf{0} \vee \exists j \leq m+1[x = j' \vee S_{m+1}(j')] \\
R^S_{m+2}(x,f) = f \quad &\leftrightarrow \quad \exists y,z \in B_m[x = (y,z)_m \wedge \mathrm{ind}(y)' = z] \\
R^S_{m+2}(x,\mathcal{P}(\omega)) = \mathcal{P}(\omega) \quad &\leftrightarrow \quad x \leq \omega \\
R^S_{m+2}(x,\mathcal{P}(e)) = \mathcal{P}(e) \quad &\leftrightarrow \quad x \leq e
\end{aligned}
$$

We define $B = \bigcup\{B_m | m \in \omega\}$ and $S = \bigcup\{S_m | m \in \omega\}$. Technically, as we define each B_m, we define a finite isomorphism of $\{0,\ldots,m\}$ to $\{0',\ldots,m'\}$. Then this isomorphism is referred to in the clause of (2-11) for $R^S_{m+2}(x,\omega)$.

This ends the main construction for this chapter.

Verification of the Desired Properties

PROPN 2.14. *The structure B of definitions (2.12,13) is an* S*-algebra.*

PROOF: We need to prove the singleton function is well-defined. We observed above that S_1 extends S_0. To prove by induction that S_{m+2} extends S_{m+1}, we assume that

S_{m+1} extends S_m. Consequently $(\cdot,\cdot)_m$ extends $(\cdot,\cdot)_{m-1}$. It will suffice to show that for any $x \in B_{m+1}$ we have $R^S_{m+2}(x,\tau) = R^S_{m+1}(x,\tau)$ for each $\tau \in T$. Then both S_{m+2} and S_{m+1} will choose the $\mathrm{ind}(x)^{th}$ atom in the same partition subset of N, i.e. the $\mathrm{ind}(x)^{th}$ atom under $\bigwedge\{R^S_{m+2}(x,\tau)\}_{\tau\in T}$.

The universal set u is trivial, because $S(x) < u = 1$ for every x, and the terms e, i and u^0 are similar to each other. For the set u^0 of non-pairs, let $x \in B_{m+1}$.

(2-12) $R^S_{m+2}(x,u^0) =\sim u^0$

$\leftrightarrow \exists y,z \in B_m\,[x = (y,z)_m]$ by defn (2-11)

$\leftrightarrow \exists y,z \in B_{m-1}\,[x = (y,z)_m]$ since $x \in B_{m+1}$

$\leftrightarrow \exists y,z \in B_{m-1}\,[x = (y,z)_{m-1}]$ by induction hypothesis

$\leftrightarrow R^S_{m+1}(x,u^0) =\sim u^0$ by defn (2-11)

For the code e of the relation E,

(2-13) $R^S_{m+2}(x,e) = e$

$\leftrightarrow \exists y,z \in B_m\,[x = (y,z)_m \wedge S_m(y) \le z]$ by defn (2-11)

$\leftrightarrow \exists y,z \in B_{m-1}\,[x = (y,z)_m \wedge S_m(y) \le z]$ since $x \in B_{m+1}$

$\leftrightarrow \exists y,z \in B_{m-1}\,[x = (y,z)_{m-1} \wedge S_{m-1}(y) \le z]$ by ind. hyp.

$\leftrightarrow R^S_{m+1}(x,e) = e$ by defn (2-11)

For the code i of the identity relation, the analysis of $R^S_{m+2}(x,i)$ is again similar.

(2-14) $R^S_{m+2}(x,i) = i$

$\leftrightarrow \exists y,z \in B_m\,[x = (y,z)_m \wedge y = z]$ by defn (2-11)

$$\leftrightarrow \exists y, z \in B_{m-1}\,[x = (y,z)_m \wedge y = z] \qquad \text{since } x \in B_{m+1}$$

$$\leftrightarrow \exists y, z \in B_{m-1}\,[x = (y,z)_{m-1} \wedge y = z] \qquad \text{by ind. hyp}$$

$$\leftrightarrow \mathrm{R}^{\mathrm{S}}_{m+1}(x,i) = i \qquad \text{by defn (2-11)}$$

The case of the term ω is somwhat different.

(2-15) $\mathrm{R}^{\mathrm{S}}_{m+2}(x,\omega) = \omega$

$$\leftrightarrow x = \mathbf{0} \vee \exists j \leq m+1\,[x = j' \vee \mathrm{S}_{m+1}(j')] \qquad \text{by defn (2-11)}$$

$$\leftrightarrow [x = \mathbf{0} \vee \exists j \leq m\,[x = j' \vee \mathrm{S}_m(j')]] \qquad \text{by ind. hyp}$$

$$\vee\, x = ((m+1)' \vee \mathrm{S}_{m+1}((m+1)'))$$

$$\leftrightarrow \mathrm{R}^{\mathrm{S}}_{m+1}(x,\omega) = \omega \vee x \notin B_{m+1} \qquad \text{by defn (2-11)}$$

The case of f is also different from the definable elements.

(2-16) $\mathrm{R}^{\mathrm{S}}_{m+2}(x,f) = f$

$$\leftrightarrow \exists y, z \in B_m\,[x = (y,z)_m \wedge \mathrm{ind}(y)' = z] \qquad \text{by defn (2-11)}$$

$$\leftrightarrow \exists y, z \in B_{m-1}\,[x = (y,z)_m \wedge \mathrm{ind}(y)' = z] \qquad \text{since } x \in B_{m+1}$$

$$\leftrightarrow \exists y, z \in B_{m-1}\,[x = (y,z)_{m-1} \wedge \mathrm{ind}(y)' = z] \qquad \text{by ind. hyp}$$

$$\leftrightarrow \mathrm{R}^{\mathrm{S}}_{m+1}(x,f) = f \qquad \text{by defn (2-11)}$$

If $\mathrm{ind}(y)' = z \in B_m$, then $\mathrm{ind}(y) \leq m$. So $\mathrm{ind}(y)'$ is defined by stage B_{m+2}. The power sets are simple.

(2-17) $\mathrm{R}^{\mathrm{S}}_{m+2}(x, \mathcal{P}(\omega)) = \mathcal{P}(\omega) \leftrightarrow x \leq \omega \qquad$ by defn (2-11)

$$\leftrightarrow \mathrm{R}^{\mathrm{S}}_{m+1}(x, \mathcal{P}(\omega)) = \mathcal{P}(\omega) \qquad \text{by defn (2-11)}$$

And in precisely the same way, we have

$$\text{(2-18)} \quad \mathrm{R}^{\mathrm{S}}_{m+2}(x, \mathcal{P}(e)) = \mathcal{P}(e) \leftrightarrow x \leq e \qquad \text{by defn (2-11)}$$
$$\leftrightarrow \mathrm{R}^{\mathrm{S}}_{m+1}(x, \mathcal{P}(e)) = \mathcal{P}(e) \quad \text{by defn (2-11)}$$

Now we have verified that $\bigwedge\{\mathrm{R}^{\mathrm{S}}_{m+2}(x,\tau)\}_{\tau\in T}$ and $\bigwedge\{\mathrm{R}^{\mathrm{S}}_{m+1}(x,\tau)\}_{\tau\in T}$ are the same. So S_{m+2} extends S_{m+1}, and $\mathrm{S} = \bigcup\{\mathrm{S}_m\}_m$ is well-defined.

The singleton function S is one-to-one, because ind is one-to-one, so each S_m is one-to-one. Its values are atoms in $^{\mathsf{N}}2$, hence also atoms in $B < {}^{\mathsf{N}}2$. Thus B is an S-algebra. ∎

PROPN 2.15. *In the structure B of definitions (2.12,13), if $\mathsf{N}^B = \{x | \mathrm{S}(x) \leq \omega\}$, then $\langle \mathsf{N}^B, E\rangle \cong \langle \omega, \in \rangle$. Moreover, the element f codes a one-to-one embedding of u into ω. Formally, the relation $\{\langle y, z\rangle | (y, z)Ef\}$ is a well-defined one-to-one function such that $(y, z)Ef \to zE\omega$.*

PROOF: The only elements of N^B are $\mathbf{0}' = \mathbf{0}$ and $(n+1)' = n' \vee \mathrm{S}(n')$, for $n \in \mathsf{N}$. Moreover, we have $j' E n'$ iff $j < n$, i.e. $j' < n'$. So $\langle \mathsf{N}^B, E\rangle \cong \langle \mathsf{N}, \in \rangle$.

For f, we see that if $(y, z)Ef$ then $z = \mathrm{ind}(y)' E \omega$. Since $z = \mathrm{ind}(y)'$, we also have that $\{\langle y, z\rangle | (y, z)Ef\}$ is well-defined and one-to-one, as required. ∎

We have left only to verify that the other five distinguished features play their intended roles. We will introduce some notation for the other clauses of (2-2) which we will use in more complex constructions to make the analysis of these elements uniform. The current construction allows us to limit the occurrence of function symbols in the terms so we only need to interpret finitely many special terms e, i, u^0, $\mathcal{P}(\omega)$ and $\mathcal{P}(e)$, together with one place predicates $v = p$ and their Boolean combinations with the special terms. These one-place predicates are interpreted

automatically since we construct Boolean algebras eventually closed under S. So interpretations of formulas such as $yEz \wedge z \neq \mathbf{0}$ will be produced via closure under S. In a more complex model in Chapter 4, these closures with others will guarantee satisfaction of a comprehension scheme for $\mathcal{L}(E,=)$.

DEFN 2.16. Let C be any set. Let $\{c_x | x \in C\}$ be a set of distinct *special constant* symbols for every $x \in C$. Let $\mathcal{L}$ be the language of e, i, u^0, ω, f (as a constant symbol, not a function symbol), $\mathcal{P}$, S, $\wedge$, $\vee$, $\sim$ and $\{c_x | x \in C\}$. An *$\mathcal{L}$-term* with special constants for C is any closed term τ in $\mathcal{L}$. A *standard $\mathcal{L}$-term* is any Boolean combination (by $\wedge$, $\vee$ and $\sim$) of e, i, u^0, $\mathcal{P}(\omega)$, $\mathcal{P}(e)$ and $\mathrm{S}(c_x)$, for any $x \in C$. The *special* $\mathcal{L}$-terms are e, i, u^0, $\mathcal{P}(\omega)$ and $\mathcal{P}(e)$. Define $T(C)$ to be the set of standard $\mathcal{L}$-terms with special constants for C. Now we recursively define an operator Def which applies to formal terms $\tau \in T(C)$ for a membership structure $\langle C, E, < \rangle$. It yields a definition for the term via an open formula of $\mathcal{L}(E, <, c_x, =)_{x \in C}$.

(2-19)

$\mathrm{Def}[e](v)$	is	$\exists v_1, v_2\,[v = (v_1, v_2) \wedge v_1 E v_2]$
$\mathrm{Def}[i](v)$	is	$\exists v_1, v_2\,[v = (v_1, v_2) \wedge v_1 = v_2]$
$\mathrm{Def}[u^0](v)$	is	$\nexists v_1, v_2\,[v = (v_1, v_2)]$
$\mathrm{Def}[\mathcal{P}(\omega)](v)$	is	$[v \leq \omega]$
$\mathrm{Def}[\mathcal{P}(e)](v)$	is	$[v \leq e]$
$\mathrm{Def}[\mathrm{S}(c_x)](v)$	is	$[v = c_x]$
$\mathrm{Def}[\sigma_1 \wedge \sigma_2](v)$	is	$[\mathrm{Def}[\sigma_1](v) \wedge \mathrm{Def}[\sigma_2](v)]$
$\mathrm{Def}[\sigma_1 \vee \sigma_2](v)$	is	$[\mathrm{Def}[\sigma_1](v) \vee \mathrm{Def}[\sigma_2](v)]$
$\mathrm{Def}[\sim \sigma](v)$	is	$[\sim \mathrm{Def}[\sigma](v)]$

Note that the singleton and power set functions must be treated differently from the others. In a non-extensional structure, a formula will define more than one object. Those objects will have distinct singletons and dominate different elements. Thus to produce a formula for the singletons and power sets, we need to introduce special constants for C which refer to specific elements of C. We will sometimes use objects as parameters in formulas, e.g. $v = p$ instead of $v = c_p$. Also we use e, i, u^0, ω, $\mathcal{P}(\omega)$ and $\mathcal{P}(e)$ as terms (constant symbols) as well as elements of C.

DEFN 2.17. We say an S-algebra *has an interpretation of* a standard $\mathcal{L}$-term τ iff there is some $y \in C$ such that $C \vDash \forall x\,[xEy \leftrightarrow \mathrm{Def}[\tau](x)]$.

Now our claim for the structure is as follows.

PROPN. 2.18. *The* S*-algebra* B *of definitions (2.12,13) has an interpretation for each special basic term* $\tau \in T$.

PROOF: We have $\mathrm{S}(x) \leq e$ iff $\mathrm{S}_m(x) \leq e$ for any and all B_m which contain x. But $\mathrm{S}_m(x) \leq e$ iff $\exists y, z\,[x = (y,z)_{m-2} \wedge \mathrm{S}_{m-2}(y) \leq z]$. That condition is equivalent to $\mathrm{Def}[e](x)$. So e is an interpretation for the membership relation as desired.

Similarly, we have $\mathrm{S}(x) \leq i$ iff $\mathrm{S}_m(x) \leq i$ for all B_m which contain x. And $\mathrm{S}_m(x) \leq i$ iff $\exists y, z\,[x = (y,z)_{m-2} \wedge y = z]$. That condition is equivalent to $\mathrm{Def}[i](x)$. So i is an interpretation for the identity relation as claimed.

We have $\mathrm{S}(x) \leq u^0$ iff $\mathrm{S}_m(x) \leq u^0$ for all B_m which contain x. And $\mathrm{S}_m(x) \leq u^0$ iff $\exists y, z\,[x = (y,z)_{m-2}]$. That condition is equivalent to $\mathrm{Def}[u^0](x)$. So u^0 contains exactly those objects which are not ordered pairs.

The power set cases are trivial. The construction definitions (2.12,13) and the formula $\mathrm{Def}[\mathcal{P}(x)](v)$ both refer directly to the order relation of ${}^{\mathsf{N}}2$ for $x = e$ or for $x = \omega$. ∎

We have distinguished the membership structure $\langle C, E\rangle$ of an S-algebra because sometimes, as in the construction to follow, it is useful to refer to the associated membership relation before we have completed closure of a structure under the operation $S = \bigcup\{S_m\}_m$. In that case, we will have a Boolean subalgebra $B_m < {}^{N}2$ and the function S_m maps each of its elements to an atom of ${}^{N}2$; but some of those atoms are outside of B_m. Still $xE_my \leftrightarrow S_m(x) \leq y$ defines a membership relation between any $x \in B_m$ and any $y \in {}^{N}2$. Even though B_m is not closed under S_m, the membership structure $\langle B_m, E_m\rangle$ does satisfy closure under Boolean operations. Upon iteration to close under singletons, that membership relation is the associated membership relation for the S so defined.

2.3. General Results

We will now present some basic results about S-algebras.

THEOREM 2.19. *The class of* S*-algebras is closed under direct limits.*

PROOF: Suppose $\psi_\alpha^\beta : C_\alpha \longrightarrow C_\beta$ are homomorphisms for all $\alpha, \beta \in D$, a directed system. Let $\psi_\alpha^\alpha = id \restriction C_\alpha$ for notational convenience. On $\{\langle\alpha, x\rangle | \alpha \in D, x \in C_\alpha\}$, define the equivalence relation $\langle\alpha, x\rangle \sim \langle\beta, y\rangle$ iff for some $\langle\gamma, z\rangle$, $\psi_\alpha^\gamma(x) = z = \psi_\beta^\gamma(y)$. Let $[\alpha, x]$ be the equivalence class of $\langle\alpha, x\rangle$, and let $C = \{[\alpha, x] | x \in C_\alpha\}$. Now $[\alpha, x] \vee [\beta, y] = [\gamma, \psi_\alpha^\gamma(x) \vee \psi_\beta^\gamma(y)]$, where $\alpha, \beta < \gamma$ and $\sim [\alpha, x] = [\alpha, \sim x]$ are well-defined. The limit system is a direct limit system of Boolean algebras, so the limit is a Boolean algebra with respect to the operations $\vee$ and $\sim$ as defined.

We now show that $S([\alpha, x]) = [\alpha, S_\alpha(x)]$ is well-defined and one-to-one, where S_α is the singleton function in C_α.

(2-20) $S([\alpha, x]) = S([\beta, y]) \leftrightarrow S_\gamma(\psi_\alpha^\gamma(x)) = S_\gamma(\psi_\beta^\gamma(y))$ for some $\gamma > \alpha, \beta$

$$\leftrightarrow \psi^{\gamma}_{\alpha}(x) = \psi^{\gamma}_{\beta}(y) \qquad \text{for some } \gamma > \alpha, \beta$$

$$\leftrightarrow [\alpha, x] \sim [\beta, y]$$

For all $\beta > \alpha$, we know $\psi^{\beta}_{\alpha}(\mathrm{S}_{\beta}(x))$ is an atom in C_{β}. So all $[\alpha, \mathrm{S}_{\alpha}(x)] \in C$ are atoms of C.

C is an initial object in the category of S-algebras. Suppose $\Phi_{\alpha} : C_{\alpha} \longrightarrow B$ for all $\alpha \in D$, and $\Phi_{\alpha} = \Phi_{\beta} \circ \psi^{\beta}_{\alpha}$ for all $\beta > \alpha$. Define $\Psi : C \longrightarrow B$ by $\Psi([\alpha, x]) = \Phi_{\alpha}(x)$. This is well-defined.

Thus, $C = \varinjlim\{C_{\alpha}\}_{\alpha \in D}$ ∎

Next we define various closure properties which are preserved in the extensional subalgebra by virtue of the fact that it is comprehensive. We will also prove that if an S-algebra is closed under these Godel-like operations then it satisfies strict quantifier-free comprehension. In Chapter 4 we will construct such an algebra. Our next result shows that that will suffice to prove the consistency of our comprehension scheme for a set theory with a universal set.

In Chapter 4 we will construct a direct limit of S-algebras which are singleton closures of finitely generated sparse Boolean subalgebras, much like the structure B of definitions (2.12,13). We will use the additional generators to make the direct limit closed under the following Godel-like operations.

DEFN 2.20. A structure $\langle C, E, \leq \rangle$ is closed under *Godel-like operations* if it satisfies all of these closure properties.

(2-21)

conjunction	$\forall a, b \exists c \forall x \, [x E c \leftrightarrow x E a \wedge x E b]$
union	$\forall a, b \exists c \forall x \, [x E c \leftrightarrow x E a \vee x E b]$
complement	$\forall a \exists b \forall x \, [x E b \leftrightarrow x \not E a]$

e codes E	$\exists e \forall x\,[xEe \leftrightarrow \exists y,z[x=(y,z) \wedge yEz]]$
i codes $=$	$\exists i \forall x\,[xEi \leftrightarrow \exists y,z[x=(y,z) \wedge y=z]]$
cartesian product	$\forall a,b \exists c \forall x\,[xEc \leftrightarrow \exists y,z\,[x=(y,z) \wedge yEa \wedge zEb]]$
relation inverse	$\forall a \exists b \forall x\,[xEb \leftrightarrow \exists y,z\,[x=(y,z) \wedge (z,y)Ea]]$
argument permutation	$\forall a \exists b \forall x\,[xEb \leftrightarrow \exists y,z,w\,[x=(y,z,w) \wedge (y,w,z)Ea]]$
argument permutation	$\forall a \exists b \forall x\,[xEb \leftrightarrow \exists y,z,w\,[x=(y,z,w) \wedge (z,w,y)Ea]]$
principal ultra-filters	$\forall a \exists b \forall x\,[xEb \leftrightarrow aEx]$
power set	$\forall a \exists b \forall x\,[xEb \leftrightarrow x \leq a]$

The *special* Godel-like operations are those other than the Boolean operations and the singleton function.

We omit closure under the range operation because Russell's paradox proves that an S-algebra with a code for E cannot be closed under ranges.

PROPN 2.21. *If C is an* S*-algebra with a code e for E, then C is not closed under formation of the range of a relation, i.e. C cannot satisfy* $\forall a \exists b[zEb \leftrightarrow \exists y[(y,z)Ea]]$.

PROOF: If we let $\mathrm{rng}(a)$ denote the range of a relation a, then C would satisfy the formula $\forall z[zE\,\mathrm{rng}(a) \leftrightarrow \exists y[(y,z)Ea]]$. In particular, if we let $r = \mathrm{rng}(\sim e \wedge i)$, then

(2-23)
$$\begin{aligned} rEr &\leftrightarrow \exists y[(y,r)E(\sim e \wedge i)] \\ &\leftrightarrow \exists y[y \not\in r \wedge y = r] \\ &\leftrightarrow r \not\in r \end{aligned}$$

This is the usual contradiction. ∎

The next theorem relates closure under the Godel-like operations to comprehension axioms. It is analogous to the Normal Form Theorem [1] which states that

closure under Godel operations implies that a transitive structure satisfies comprehension for axioms with bounded quantification. Our theorem is essentially simpler since we do not consider quantification. We only need to point out that consideration of puf, the one extra Godel-like operation we use, is a simple matter. Of course, in our construction, the elements of C are not genuine sets, but we will still refer to them as "interpretation/extension of F in the structure C" and we use the notation F^C for $\{\vec{x}|\, F(\vec{x})\}^C$, an element of C. (We will later define $\mathrm{extn}_C(x)$ to be the *set* of objects in C which are C-elements of x, i.e. $\mathrm{extn}_C(x)$ is the set $\{\vec{y}|\, F(\vec{y})\}$, an element of V.) Consistent with [1], we take n-tuples to be ordered pairs grouped to the left, i.e. $(x_1,\ldots,x_{n+1}) = ((x_1,\ldots,x_n),x_{n+1})$.

Since most of our structures will be non-extensional, certain convenient expressions will require some abuse of notation. In the intermediate structures, our concern for the most part is *existence* of such objects as "the Cartesian product of sets a and b", and we will use notations such as $a\times b$, but these objects will not be unique as the terminology and notations imply. To be fully correct, we could choose elements from extensional equivalence classes as values of a single-valued Skolem function $a\times b$. The "Skolem functions" would only be partial functions until closure under the operations is achieved in the limit. Rather than use terms which have no interpretation, we will use terms with non-unique interpretions. In the end, we arrange uniqueness when we pass to an extensional substructure which preserves the desired closure properties. In that extensional substructure of the limit, our terminology is fully correct. In all structures, $\wedge$, $\vee$, $\sim$ and S are genuine single-valued functions.

To allow parameters in the comprehension axioms, we need to consider one place atomic formulas $F(v)$ which are $p = v$, $v = p$, vEp or pEv. We use a simple Boolean algebra which contains some "free generators" as in previous constructions. For atomic formulas $p = v$ or $v = p$, there is no problem since the extension of either

formula is just the singleton of p, and we will have closure under singletons as always. For formulas vEp, where p is a parameter we have already constructed, p itself serves as the extension of the formula. The only non-trivial case of atomic formulas with parameter p is pEv, where p is a parameter. This will require closure under an operation not analogous to those used in construction of L. For any given element p, the limit structure will contain the principal ultrafilter generated by the singleton of p, which we will denote by $\mathrm{puf}(c_p)$ or simply $\mathrm{puf}(p)$. That element will serve as the extension of the formula pEv. Recall that closure of L under Godel operations implies that L satisfies comprehension for formulas with bounded quantification. By a simple variation of that proof, once we establish that the limit of the system we construct is closed under our Godel-like operations we will be able to prove that the limit satisfies a comprehension scheme which we now define.

In Jech's proof of the analogous theorem, which is concerned with constructibility in ZF, the normal form of formulas he defined for that proof requires that no variable be repeated in an atomic subformula. In other words, it is exactly the notion of a strictly quantifier-free formula.

THEOREM 2.22. *If an* S*-algebra* C *is closed under the operations* $\times$, inv, π_{23}, π_{123} *and* $\mathrm{puf}(c_p)$, *with* e *and* i *which code* E *and* $=$, *respectively, then the associated membership structure* $\langle C, E\rangle$ *satisfies strictly quantifier-free comprehension with parameters for* $\mathcal{L}(E,=)$.

PROOF: Let C be any S-algebra closed under these operations. For any strictly quantifier-free formula $F(\vec{p},\vec{v})$, we must show that there is an element y of C such that we have $\langle C, E\rangle \vDash \forall\vec{p}\forall\vec{x}\,[\vec{x}Ey \leftrightarrow F(\vec{x},\vec{p})]$ We use induction on subformulas of F. The complex case is the case of the atomic formulas since we allow "dummy"

variables, i.e. the formulas $F(v_1,\ldots,v_5)$ and $F(v_1,\ldots,v_6)$ are considered separately even if both are v_3Ev_1.

We will use induction on the number of variables in atomic formulas and on the complexity of non-atomic formulas. We will show that the extension of an atomic formula without dummy variables or parameters is constructible with operations directly associated with membership and identity predicates, together with simple applications of the inverse operator. If F is an atomic formula *with* dummy variables, e.g. if $F(v_1,\ldots,v_5)$ is v_3Ev_1 we take care of it by those same relations with applications of Cartesian product with u (the unit of the Boolean algebra), and relation inverse and argument permutation operations. Finally, the extension of a non-atomic quantifier-free strictly quantifier-free formula is constructed from the above relations via the Boolean operations.

The fact that we allow a parameter p in comprehension axioms requires some additional considerations. If the parameter does *not* occur in the formula, e.g. $F(v_1,\ldots,v_5)$ is v_2Ev_1 and $G(v_1,\ldots,v_4)$ is $F(v_1,\ldots,v_4,p)$, then the required element is constructed when we take v_2Ev_1 as $H(v_1,\ldots,v_4)$. If a parameter p actually does occur in an atomic formula, in the cases $v = p$ and $p = v$, the extension of a variation with dummy variables is constructible from $\mathrm{S}(p)$ by Cartesian product with u, relation inverse and argument permutations. Aside from those simple cases, we have four additional cases of atomic formulas. They are pEv or vEp as $F(v)$ and pEv or vEp as $F(v_1,\ldots,v_k)$ for $k > 1$. The extension of pEv is exactly what $\mathrm{puf}(c_p)$ gives us. And the extension of vEp is simply p itself. Again, in the case of variants of each formula with dummy variables, the extension is constructible by Cartesian product with u, relation inverse and argument permutations. Of course, comprehension axioms require that we have an open formula, so we will have at least one variable.

Case Ia: Suppose $F(v_1, \ldots, v_n)$ is $v_j = p$, $p = v_j$, pEv_j or v_jEp.

If $n = 1$ then F must be one of $v_j = p$, $p = v_j$, pEv_j or v_jEp. In these cases, for F^C we choose S(p), puf(p) or p itself, as discussed in the first pages of this chapter. If $n > 1$ then one of the those elements, S(p), puf(p) or p, generates F^C when we take its Cartesian product with $n - 1$ factors u; specifically $(((u^{j-1} \times x) \times u) \cdots \times u)$, where x is one of $S(p)$, puf(p) or p.

Case Ib: Suppose $F(v_1, \ldots, v_n)$ is v_jEv_k or $v_j = v_k$.

This is the technically complicated case. We argue by induction on the number n of variables. Cases where a parameter p occurs in the atomic formula were separated in case Ia. So here we have a parameter-free, strictly quantifier-free atomic formula. We will uniformly use v_j and v_k to denote the two distinct variables that are not dummy variables.

If $n = 2$ we have these three possibilities. Either F is v_1Ev_2 with $F^C = e$; or F is v_2Ev_1 with $F^C = \text{inv}\, e$; or F is $v_1 = v_2$ (or $v_2 = v_1$) with $F^C = i$.

For induction on the number of variables, assume that we have the extension of any strictly quantifier-free formula with less than n variables. To consider E and $=$ simultaneously, let R be either E or $=$.

If $n > 2$, and $j, k \neq n$, we have $x = \{(v_1, \ldots, v_{n-1}) | \, v_jRv_k\}^C$ by the induction hypothesis, since the formula v_jRv_k as $G(v_1, \ldots, v_{n-1})$ has less than n variables. Then $x \times u$ is the extension in C of F, since the last variable is not restricted, i.e. $\{(v_1, \ldots, v_n) | \, v_jRv_k\}^C$.

If $n > 2$, $j, k \neq n - 1$, then to arrange the previous situation we exchange the last variable with the unrestricted dummy variable next to the last. By the induction hypothesis in the case immediately above, where $n > 2$, and $j, k \neq n$, we have $x = \{(v_1, \ldots, v_{n-2}, v_n, v_{n-1}) | \, v_jRv_k\}^C$. Then when we exchange the last two variables we have $\pi_{23}(x) = \{(v_1, \ldots, v_{n-2}, v_{n-1}, v_n) | \, v_jRv_k\}^C$.

In the other cases, v_j and v_k are the last two variables v_{n-1} and v_n. There are only two possibilities, since v_j and v_k are not the same variable.

If $n > 2$, $j = n-1$, $k = n$ we have $\{(v_{n-1}, v_n) | \, v_{n-1} R v_n\}^C$ as $x = i$ if R is $=$,and $x = e$ if R is E, since there are only two variables and their names do not matter. Then, we can use u^{n-2} to allow for the dummy variables. An application of π_{123} brings this factor to the left, where it must be to form a set of n-tuples. Formally, we have x such that $x \times u^{n-2} = \{((v_{n-1}, v_n), (v_1, \ldots, v_{n-2})) | \, v_{n-1} R v_n\}^C$, and

$$\pi_{123}(x \times u^{n-2}) = \{(v_1, \ldots, v_{n-2}, v_{n-1}, v_n)) | \, v_{n-1} R v_n\}^C \tag{2-24}$$

If $n > 2$, $j = n$, $k = n-1$ we do a similar construction with the inverse operation. So we have x such that $x \times u^{n-2} = \{((v_{n-1}, v_n), (v_1, \ldots, v_{n-2})) | \, v_n R v_{n-1}\}^C$, and

$$\pi_{123}(\mathrm{inv}\, x \times u^{n-2}) = \{(v_1, \ldots, v_{n-2}, v_{n-1}, v_n)) | \, v_n R v_{n-1}\}^C \tag{2-25}$$

This finishes the atomic formulas. We proceed with the other cases, which are handled automatically by the Boolean structure.

Case II: Suppose $F(v_1, \ldots, v_n)$ is $\neg H(v_1, \ldots, v_n)$.
Some element x is $\{(v_1, \ldots, v_n) | \, H(v_1, \ldots, v_n)\}^C$ Then $(u^n)^C \wedge (\sim x)$ is

$$\{(v_1, \ldots, v_n) | \, F(v_1, \ldots, v_n)\}^C \tag{2-26}$$

Case III: Suppose $F(v_1, \ldots, v_n)$ is $H_1(v_1, \ldots, v_n) \wedge H_2(v_1, \ldots, v_n)$.
If elements x_j are $\{(v_1, \ldots, v_n) | \, H_j(v_1, \ldots, v_n)\}^C$ for $j = 1, 2$ respectively, then $x_1 \wedge x_2$ is $\{(v_1, \ldots, v_n) | \, F(v_1, \ldots, v_n)\}^C$.

Case IV: Suppose $F(v_1,\ldots,v_n) = H_1(v_1,\ldots,v_n) \vee H_2(v_1,\ldots,v_n)$. Similarly, if elements x_j are

$$\{(v_1,\ldots,v_n)|\, H_j(v_1,\ldots,v_n)\}^C \tag{2-27}$$

for $j = 1,2$ respectively, then $x_1 \vee x_2$ is $\{(v_1,\ldots,v_n)|\, F(v_1,\ldots,v_n)\}^C$ ∎

Any structure closed under Godel-like operations contains an element e which is (a code for) its membership relation, because it satisfies comprehension for the formula vEw. We denote the complement of the empty element in such structures by u. Then, since we have closure under ordered pairs, any such structure contains (u,e), which codes the structure itself. Such structures also contain $\mathrm{puf}((u,e))$ which is the "set" of all properties of the structure, as interpreted in this model. Not all properties are necessarily definable, but if a satisfaction relation is definable then the truth set of (u,e), in some sense, would be computable relative to $\mathrm{puf}((u,e))$. Thus the operator puf has some interest other than simply to allow parameters in our formal axioms.

CHAPTER 3

COMPREHENSIVE EXTENSIONAL S-SUBALGEBRAS

The main result in this chapter is that from any given countable S-algebra C one can extract an extensional S-subalgebra B such that if x is an arbitrary element of the S-algebra C, there is an element y in the S-subalgebra B which contains exactly the same elements of the S-subalgebra B that x does. Consequently, this S-subalgebra preserves closure under Gödel-like operations. As we saw in Chapter 2, these closure properties imply that the extensional substructure satisfies strictly quantifier-free comprehension.

The properties preserved in the substructure tend to be those properties of the larger structure which relate to the S-algebra operations S, $\wedge$, $\vee$ and $\sim$. In particular, the substructure B preserves the role of an element which is (a code for) the membership relation, the singleton function or the complement function, if such elements exist in the larger structure C.

The extensional substructure B preserves closure under cartesian products, which our construction of Chapter 4 will satisfy. It does not necessarily preserve the role of an element which is (a code for) the cartesian product function. Suppose there is a single-valued function $\cdot\times\cdot$ over C which satisfies the definition of the cartesian product function. Further suppose an element p of C contains all $(x, y, x\times y)$ for x and y in C. For any x and y in B there will be some $z = x\times y$ in C and some z' in B which represents z. Typically $z' \neq z$. But since $z' \neq z$ and the element p represents a single-valued function, then (x, y, z) is not in B and (x, y, z') is not an E-member of p. Such relations will not be obviously preserved in the extensional

substructure unless the representation p in C contains *all* (x, y, z') such that $z' \in C$ is extensionally equivalent to z.

The problem with (a code for) the cartesian product function as an element is not avoided if we restrict our attention to more narrowly definable sets. It arises because two cartesian products can differ by finite extension. The difference between $S(0) \times u$ and $S(0) \times (\sim S(0))$ is only one element, $(0, 0)$. This difficulty is perhaps surprising, since cartesian product closure and an element which codes the cartesian product function are both fairly easy features to achieve. But as noted in the previous paragraph, the problem is actually with the single-valued nature of the cartesian product relation. This adds some support to the notion that single-valued relations such as ZF uses in its replacement axiom, really are not generally more constructible than arbitrary relations. Limitation of complexity is the relevant issue, not limitation of size.

We can construct an extensional membership structure as a quotient of an S-algebra. One can simply define an equivalence relation by extensional equivalence, construct the quotient, and iterate transfinitely. If we start with a countable algebra, then the transfinite iteration stabilizes before $\aleph_1$ iterations. That method has been previously investigated in [14], in which a general preservation theorem for the quotient is proven. In [14], Hinnion carries out the construction of an extensional quotient for a structure with an *arbitrary* binary relation E on a structure of arbitrary cardinality. He cannot construct a retraction of the quotient to a subalgebra, since that is not possible in such generality.

To follow such a construction to a more useful result, we need to have some assumptions that make E more distinctly like a set membership relation. We will not use this method. One of the elements we will want in our structures is the cartesian product $u^2 = u \times u$. A projected image of u^2 does not preserve the character of u^2. If

the original structure contains empty objects and is closed under ordered pairs, then pairs of those empty objects are elements of u^2. With Weiner-Kuratowski codes for ordered pairs, the quotient identifies those fundamentally empty ordered pairs with the empty set, which is not an ordered pair. Thus, the image of u^2 and the image of its complement are not disjoint. So the preservation theorem for quotients of an arbitrary binary relation simply is not strong enough.

In addition to the Boolean algebra structure of S-algebras, we also use the very distinctive feature of Boolean algebras that every countable Boolean algebra is projective [15]. It follows by the countable axiom of choice that every quotient of a countable Boolean algebra has a retraction. We will detail the construction of such a retraction because given an S-algebra C and its Boolean reduct, we will want to have a generator tree for the retraction Q of various quotients of the Boolean reduct. We will use such a tree to thin out the S-algebra in a transfinite iteration. The result will be an extensional S-subalgebra rich enough to have a useful representative of each element of the original algebra.

We will construct countable Boolean subalgebras of a complete atomic Boolean algebra with countably many atoms. Since every S-algebra is atomic and has infinitely many atoms, the Boolean completion of any countable S-algebra is isomorphic to ${}^{\mathsf{N}}2$. So without loss of generality we will take the complete Boolean algebra to be ${}^{\mathsf{N}}2$. Again, the notion of atom is relative to the Boolean subalgebra. To avoid confusion we will only rarely and very explicitly use the term "atom" to refer to anything other than atoms $\chi_{\{j\}}$ of ${}^{\mathsf{N}}2$.

Now we define some notation related to the way we construct a finite sparse Boolean subalgebra of ${}^{\mathsf{N}}2$.

DEFN 3.1. If Q is a Boolean algebra, then $G \subseteq Q$ *generates* Q *by sups* if we have

$Q = \{0\} \cup \{g_1 \vee \cdots \vee g_n | g_j \in G\}$. A *generator tree* G for Q is any binary tree with root $\mathbf{1}$ which generates Q by sups and whose initial segments are finite. For any generating subtree H of G, and for any $g \neq \mathbf{1}$ in G, let $\mu_H(g)$ be the least h in H strictly greater than g. Since we define any generating tree H to have root $\mathbf{1}$, the value of $\mu_H(g)$ is defined for all generating subtrees H and for all $g \neq \mathbf{1}$. Note that $\mu_G(g)$ is the *parent* of g in G. We will call $\mu_G(g) \backslash g$ the *sibling* of g in G.

We now introduce a property of S-subalgebras which is sufficient to preserve the features of interest in this work.

DEFN 3.2. Suppose C is an S-algebra and $B < C$ is an S-subalgebra. For every x in the larger algebra C, define the *B-extension* of x to be $\mathrm{extn}_B(x) = \{y \in B | \mathrm{S}(y) < x\}$. B is a *comprehensive* S-*subalgebra of* C iff for every $x \in C$ there is some $x' \in B$ which contains the same elements of B as x does, i.e. such that $\mathrm{extn}_B(x') = \mathrm{extn}_B(x)$. Suppose $C < {}^{\mathbf{N}}2$ is countable and is enumerated as $\{x_j\}_{j \in \omega}$ without repetitions. Again, we will use an index function $\mathrm{ind} : C \xrightarrow{1-1} \omega$ defined by $\mathrm{ind}(x_j) = j$.

LEMMA 3.3. *Suppose C is a countable Boolean subalgebra of ${}^{\mathbf{N}}2$ and $I \subset C$ is an ideal. Then there is a retraction r which maps C/I onto a subalgebra $Q < C$ which has a generator tree $G \subset C$. If I contains all the elements of C which have finite support, then any retraction of C/I is sparse.*

PROOF: Enumerate $C = \{x_j | j \in \omega\}$ with no repetitions. Let $G_0 = \{\mathbf{1}\}$. We will successively consider the elements x_n to construct the tree G_{n+1} which will end-extend G_n with the additional leaves necessary to regenerate x_n by finitary Boolean operations and elements of I. To accomplish this, we will add the parts of x_n which g separates from x_n, i.e. $g \wedge x_n$ and $g \backslash x_n$, so long as both parts are outside of I. So

$$G_{n+1} = G_n \cup \{g \wedge x_n, g \backslash x_n | \text{g is a leaf of } G_n; \text{ both } g \wedge x_n, g \backslash x_n \notin I\} \tag{3-1}$$

Observe that G_{n+1} is a finite tree and an end-extension of G_n. Its incomparable elements are disjoint, and each element $g \neq \mathbf{1}$ differs from its parent by more than an element of I, i.e. $\mu_G(g) \backslash g \notin I$. Then $G = \bigcup\{G_n\}_n$ is a binary tree with root $\mathbf{1}$ and with finite initial segments.

Let $Q = \{0\} \cup \{g_1 \vee \cdots \vee g_n | g_j \in G\}$. Trivially, Q is closed under unions.

(3-2)
$$\left(\bigvee_j g_j\right) \wedge \left(\bigvee_k g'_k\right) = \bigvee_{j,k} (g_j \wedge g'_k) = \bigvee\{\min\{g_j, g'_k\} | g_j, g'_k \text{ are comparable}\}$$

which is in Q. So Q is closed under intersections.

Q is closed under complements. Since $\sim \left(\bigvee_j g_j\right) = \bigwedge_j (\sim g_j)$, consider some g_j in some G_n.

(3-3)
$$\sim g_j = \bigvee\{g \in G_n | g, g_j \text{ incomparable}\}$$

which is in Q. So Q is the Boolean algebra generated by G.

If $[x]$ denotes the equivalence class of x in C/I, define $r\colon C/I \xrightarrow{\simeq} Q$ by

(3-4)
$$r[x_n] = \bigvee\{g \in G_{n+1} | g \text{ is a leaf and } g \wedge x_n \notin I\}$$

First notice $r[x_n] \equiv x_n$, since

(3-5)
$$x_n = \bigvee\{g \wedge x_n | \ g \text{ is a leaf of } G_{n+1}\} = (r[x_n] \backslash y) \vee z, \text{ where}$$

(3-6)
$$y = \bigvee\{g \backslash x_n | g \text{ is a leaf of } G_n, \text{ and } g \backslash x_n \in I\}, \text{ and}$$

(3-7) $$z = \bigvee\{g \wedge x_n | \, g \text{ is a leaf of } G_n \text{ and } g \wedge x_n \in I\}$$

If $r[x] \equiv r[y]$ then $r[x] = r[y]$. Each of $r[x]$ and $r[y]$ is a sup of selected disjoint leaves from G_{n+1}, where n is the larger of the indices $\mathrm{ind}(x)$ and $\mathrm{ind}(y)$. If $r[x] \neq r[y]$, then we have some leaf g in G_{n+1} with $g \leq r[x]$, say, and $g \wedge r[y] = 0$. Since $g \notin I$, it follows that $r[x] \not\equiv r[y]$.

The map r is a homomorphism because

(3-8) $$\begin{aligned} [x] \vee [y] = [z] &\to r[x] \vee r[y] \equiv r[z] \qquad \text{since } r[w] \equiv w \\ &\to r[x] \vee r[y] = r[z] \qquad \text{since, if } v \equiv w, \text{ then } v = w \end{aligned}$$

Similarly, we have $[\sim x] = [y] \to r[\sim x] \equiv r[y] \to r[\sim x] = r[y]$.

The map r is well-defined since

(3-9) $$x \equiv 0 \to r[x] = \bigvee \emptyset \to r[x] = 0 = r[0]$$

Since incomparable elements of G are disjoint, we know $r[g] = g$ for all $g \in G$. Then $r[x] = x$ for all $x \in Q$, and r maps onto $\mathbf{Q}$.

To show r is one-to-one, suppose that $x \not\equiv 0$. Then for every $n \in \omega$, we have $\bigvee\{g \in G_n | \, g \text{ is a leaf}\} = 1$. If $x = x_n$, then $x \wedge 1 = \bigvee\{g \wedge x | \, g \text{ is a leaf of } G_{n+1}\}$. Some $g \wedge x$ is not in I, since x is not in I. So $r[x] \neq 0$ and r is one-to-one.

Finally, suppose that I contains all the elements of C which have finite support; then any retraction r of C/I must be sparse. Suppose $r[x] \neq 0$. Then $x \not\equiv 0$. Hence $r[x] \equiv x \not\equiv 0$. It follows that $r[x]$ has infinite support. ∎

We will use a variation of the conjunctive normal form theorem.

DEFN 3.4. Suppose C is a Boolean algebra with an ideal I and a retraction Q of C/I. For any x in Q, a *Q-normal form* for x is $(q\backslash y) \vee z$ where $q \in Q$ and $y, z \in I$ with $y \leq q$ and $z \wedge q = \mathbf{0}$. If C is an S-algebra with singleton function S, the S-*closure* of any Boolean subalgebra Q of C, denoted $\mathrm{cl}_S(Q)$, is the smallest S-subalgebra of C which contains Q and is closed under singletons.

For example, $\{\mathbf{0}, \mathbf{1}\}$ is a sparse subalgebra of C, and its closure under any singleton function S is the minimum S-algebra M. A more significant example is the S-closure of a retraction Q of the quotient of a countable $C < {}^{\mathbf{N}}2$, where the ideal of the quotient is the set of finitely supported elements in C. In this case the parts y and z in the Q-normal form for an element x will be finitely supported. Then for many purposes, they can be replaced by finitely many singletons. The infinite part q is in the retraction. Thus we can represent the extension of x by an element of the subalgebra $B = \mathrm{cl}_S(Q)$.

PROPN (Q-NORMAL FORMS) 3.5. *Suppose $C < {}^{\mathbf{N}}2$ is a countable Boolean subalgebra, and $I \subset C$ is an ideal. Let Q denote a retraction of C/I. Then for any x in C there are unique $q \in Q$ and $y, z \in I$ such that $x = (q\backslash y) \vee z$, with $y \leq q$ and $z \wedge q = \mathbf{0}$.*

PROOF: Let $r : C/I \xrightarrow{\sim} Q$ be a retracting map. For $q = r[x]$, let $y = q\backslash x$ and $z = x\backslash q$. Then $y, z \in I$, since $r[x] \equiv x$ for any retraction r of $C\backslash I$. Clearly, we have $x = (q\backslash y) \vee z$.

We show the uniqueness of q, y and z. Suppose $(q'\backslash y') \vee z' = x$ with q', y' and z' as required for a normal form. Then $q' = q$. Otherwise, say $q\backslash q' \notin I$. By routine

Boolean algebra we have

$$(3\text{-}10) \qquad [(q\backslash y) \vee z]\backslash[(q'\backslash y') \vee z'] = [(q\backslash y) \vee z] \wedge \sim [(q'\backslash y') \vee z']$$
$$\geq (q\backslash y) \wedge \sim (q'\backslash y') \wedge \sim z'$$
$$\geq (q \wedge \sim y) \wedge \sim q' \wedge \sim z'$$
$$= (q\backslash q')\backslash(y \vee z')$$

This would not be an element of I, since $(q\backslash q') \wedge (y \vee z') \in I$ and we have that $(q\backslash q') = [(q\backslash q')\backslash(y \vee z')] \vee [(q\backslash q') \wedge (y \vee z')]$. Thus the two expressions for x would have a non-zero difference, a contradiction.

Suppose $y\backslash y' \neq \mathbf{0}$. Then, since z' is disjoint from q, we know $x = (q\backslash y) \vee z$ fails to dominate some atom which is dominated by $x = (q\backslash y') \vee z'$, a contradiction. So we have $y' = y$.

Similarly, we have $z = z'$. Suppose $z'\backslash z \neq \mathbf{0}$. Then $(q\backslash y) \vee z' = x$ dominates some atom not dominated by $(q\backslash y) \vee z$, a contradiction. ∎

If we have an S-algebra, and I is the ideal of elements of finite extension, then we can recapture the extension of x if we replace y and z in the normal forms with finitely many singletons, as follows.

DEFN 3.6. Suppose $C < {}^{\mathbf{N}}2$ is an S-algebra with a countable Boolean quotient whose ideal I extends the ideal of elements of finite support. Let r be a retraction of this quotient onto $Q < C$, and let $B = \mathrm{cl}_{\mathrm{S}}(Q)$. If $x \in C$ has Q-normal form $(x'\backslash x_0) \vee x_1$, let $y_j = x_j \wedge \bigvee \mathrm{S}[B]$. We define the *interpretation of x in B* to be $x^B = (x'\backslash y_0) \vee y_1$.

We will apply this notion where I is simply the ideal of finitely supported elements. Notice that $q \in B$. Since the elements of I have finite support, each of y_0 and y_1 are sups of finitely many singletons of B. Thus, the interpretation of x in B is an

element of B. Also, $\mathrm{extn}_B(x^B) = \mathrm{extn}_B(x)$, so B is a comprehensive S-subalgebra of C.

Next we describe the succession step in the transfinite recursive construction of an extensional comprehensive S-subalgebra. We will apply it to countable atomic Boolean algebras. The elements of such algebras have Q-normal representations with an element q not in I, where q is generated by sups from a tree G. That element q is modified by addition and subtraction of finitely many singletons. In the next lemma we observe that this comprehensive subalgebra is almost extensional.

LEMMA 3.7. *Let $C = \mathrm{cl}_S(Q)$ be an S-algebra included in ${}^{\mathbf{N}}2$, where Q is a sparse Boolean subalgebra generated by sups from a generating tree G. There is a generating subtree $G' \leq G$ which generates $Q' \leq Q$ by sups such that, if $C' = \mathrm{cl}_S(Q')$, then*

(i) *C' is a comprehensive S-subalgebra of C.*

(ii) *$\forall x, y \in C'[x \neq y \rightarrow extn_C(x) \neq extn_C(y)]$*

PROOF: We will prune from G the nodes g which have finite C-extension. If we start with such a node and trace upward to a maximal such node m, the sibling s of m has infinite C-extension, but it is extensionally representable via the common parent p of m and s. We delete s with m and the descendants of m, but not necessarily the descendants of s. If the sibling s has two children with infinite C-extension, then we keep them, and they become the two children of p in our new binary tree. If s had only one C-infinite child, then we follow a finite path of unique children with infinite C-extension. This path ends in one of two ways. If we find a node which has two C-infinite children, then they are the two children of p in the new tree. On the other hand we may find a leaf l with infinite C-extension. Since this path is finite and each node was co-finite in its parent, with respect to C, we know that l is co-finite in p. Then we can also delete l and represent it extensionally via p. So after thinning G,

the node p either has become a leaf or has acquired exactly two new children. Thus our new tree is still a binary generator tree with root $\mathbf{1}$. This procedure will never delete the root, which has infinite extension in any S-subalgebra, since S is one-to-one. The result of this thinning is that we delete any node with finite C-extension and we delete the sibling of each such node. Formally, let

$$G' = \{\mathbf{1}\} \cup \{g \in G \backslash \{\mathbf{1}\} |\ \mathrm{extn}_C(g) \text{ and } \mathrm{extn}_C(\mu_G(g) \backslash g) \text{ are infinite}\} \tag{3-11}$$

A sparse Boolean subalgebra $Q' \leq Q$ is generated by sups from G'.

To prove (i), let $x \in C$. We have the Q-normal representation $x = (q \backslash y) \vee z$, and $q = g_1 \vee \cdots \vee g_n$ for some $g_1, \ldots, g_n \in G$. We will discard any g_j with finite C-extension, which is then not in G'. We want to replace any other g_j not in G' by $g'_j = \mu_{G'}(g_j)$. Over C, the old generator g_j is co-finite in g'_j. So to reconstruct the extension of g_j, we only need to subtract the finitely many singletons of spurious elements which are introduced when we replace such g_j with their near ancestors g'_j. So let

$$q' = \bigvee \{g_j |\ g_j \in G'\} \vee \bigvee \{\mu_{G'}(g_j) |\ g_j \notin G' \text{ and } \mathrm{extn}_C(g_j) \text{ is infinite}\} \tag{3-12}$$

Then $q' \in Q'$. Over C, we know q' differs by finite extension from q. Over C, it differs only finitely from $x = (q \backslash y) \vee z$. Both $y' = (q' \backslash x) \wedge \bigvee \mathrm{S}[C']$ and $z' = (x \backslash q') \wedge \bigvee \mathrm{S}[C']$ are sups of finitely many singletons in C'. So they are in C' and $(q' \backslash y') \vee z'$ is in C'. We have

$$\mathrm{extn}_C((q' \backslash y') \vee z') = \mathrm{extn}_C(x) \tag{3-13}$$

Thus C' is a comprehensive subalgebra of C.

To prove (ii), suppose $x_j \in C'$ for $j = 1, 2$. Let

(3-14) $$x_j = (q_j \backslash y_j) \vee z_j \text{ for } j = 1, 2$$

be their Q'-normal representations in C', where $q_j \in Q'$. Then

(3-15)

$$q_1 \neq q_2 \Longrightarrow \text{extn}_C(q_1) \text{ and } \text{extn}_C(q_2) \text{ differ by infinitely many elements}$$
$$\Longrightarrow \text{extn}_C(x_1) \text{ and } \text{extn}_C(x_2) \text{ differ by infinitely many elements}$$

Suppose that either $q_1 = q_2$ with $y_1 \neq y_2$, or $q_1 = q_2$ with $y_1 = y_2$ and $z_1 \neq z_1$. Then $\text{extn}_C(x_1) \neq \text{extn}_C(x_2)$, since

(3-16) $$\exists z \in C' \subseteq C\, [\mathrm{S}(z) \leq x_1 \nleftrightarrow S(z) \leq x_2]$$

.

Thus, in any case, then $\text{extn}_C(x_1) \neq \text{extn}_C(x_2)$. ∎

THEOREM 3.8. *Suppose an* S*-algebra* $C < {}^{\mathbf{N}}2$ *is* $\mathrm{cl}_{\mathrm{S}}(Q)$ *for some sparse* Q*, where* Q *is generated by sups from a tree* G*. Then there is an* S*-subalgebra* B *of* C *such that*

(i) B *is extensional*

(ii) B *is a comprehensive subalgebra of* C.

PROOF: Let $C_0 = C$ and define $C_{\alpha+1} = C'_\alpha$ as in Lemma 3.7, with G'_α and Q'_α also as in Lemma 3.7. For a limit ordinal λ, we define $G_\lambda = \bigcap\{G_\alpha\}_{\alpha<\lambda}$. Let Q_λ be the Boolean algebra generated by sups from G_λ and let $C_\lambda = \mathrm{cl}_{\mathrm{S}}(Q_\lambda)$. Since C is

countable, there must be some $\alpha < \aleph_1$ such that $C_{\alpha+1} = C_\alpha$, and we let $B = C_\alpha$. Then B is closed under singletons. Let $x, y \in C_{\alpha+1}$.

(3-17)

$$x_1 \neq x_2 \Longrightarrow x_1 \text{ and } x_2 \text{ are extensionally different over } C_\alpha$$
$$\Longrightarrow x_1 \text{ and } x_2 \text{ are extensionally different over } C_{\alpha+1}$$

Thus B is extensional.

Now for all $x \in C$ and $\beta < \aleph_1$, there is $y \in C_\beta$ such that the C_β-extension of y equals the C_β-extension of x. Suppose for induction that this claim is true for all $\gamma < \beta$. Then if $\beta = \gamma + 1$, the claim is also true for β by Lemma 3.7.

This same claim holds for a limit ordinal β. Suppose x has a Q_0-normal representation $(q \backslash y) \vee z$ where $q = g_1 \vee \cdots \vee g_n$. Further suppose g_j is finite over C_β, or if, over C_β,the generator g_j is co-finite in its nearest ancestor g_j' in G_β. Then we can reconstruct the C_β-extension of g_j with $g_j' \in C_\beta$ and finitely many singletons. So such a g_j is not problematic.

Suppose g_j and $g_j' \backslash g_j$ are infinite over C_β, where g_j' is the least upper bound of g_j in G_β. Then g_j is infinite over C_γ for all $\gamma < \beta$. The problem is that the least ancestor g_j'' of g_j in G_γ may be smaller than $g_j' \in G_\beta$ for $\gamma < \beta$. But a finite path in $G_0 = G$ connects g_j to g_j'. The nodes in this path are all infinite over C_β and all preceding C_α. Since they are deleted in thinning G_0, we know each element in the path differs finitely from its parent in G_0 over some C_γ with $\gamma < \beta$. So each element differs finitely from its parent in G_0 over C_β, which is smaller than each C_γ with $\gamma < \beta$. Thus g_j is co-finite in $g_j' \in G_\beta$ over C_β.

So let

(3-18) $q' = \bigvee\{g_j \in G_\beta | g_j \text{ is infinite over } C_\beta\}$

$$\vee \bigvee\{g'_j \in G_\beta | g_j \notin G_\beta \text{ and } g_j \text{ is infinite over } G_\beta\}$$

Then q' is in Q_β. There are only finitely many g_j. Only the g_j which are finite over C_β are simply deleted. Those $g_j \in G_\beta$ which are infinite over C_β are retained. Those $g_j \notin G_\beta$ which are infinite over C_β are replaced by $g'_j \in G_\beta$, which are only finitely larger than g_j over C_β. Thus q' differs only finitely from q and from x over C_β. Let $y' = (q'\backslash x) \wedge \bigvee S[C_\beta]$ and $z' = (x\backslash q') \wedge \bigvee S[C_\beta]$. Then y' and z' are sups of finitely many singletons in C_β and $x' = (q'\backslash y') \vee z'$ has C_β-extension equal to that of x.

The tree G_β generates Q_β by sups and has root $\mathbf{1}$ with only finite initial segments. So the induction hypothesis is preserved and the transfinite recursion continues for all α. ∎

Our main result in this chapter is that the substructure of Theorem 3.8 preserves closure under Gödel-like operations. We also have interest in preservation of some other features as discussed in the Theorem 3.9.

THEOREM 3.9. *Suppose C is an S-algebra which is closed under (special) Gödel-like operations. If B is an extensional comprehensive S-subalgebra of C, then B is closed under Gödel-like operations. If C is closed under the projections $\pi_j((x_1, x_2)) = x_j$, then B is also closed projections. If C contains (codes for) the projections as elements, then B contains (codes for) them.*

PROOF: The proof is quite straightforward. Of course closure of an S-algebra under the special Gödel-like operations implies closure under all Gödel-like operations.

C contains e and i which are (codes for) E and $=$. It follows that B contains e' and i' which are (codes for) the restrictions of E and $=$ to B.

To prove cartesian product closure for B, suppose $a, b \in B$. So there is some $c \in C$ such that for all $x \in C$, we have xEc iff there are $y, z \in C$ such that $x = (y, z)$ with yEa and zEb. Then for some $c' \in B$ and all $x \in B$, we have xEc' iff there are $y, z \in B$ such that $x = (y, z)$ with yEa and zEb. So c' is the witness of cartesian product closure in B for $a, b \in B$.

B is closed under inverses. For any $a \in B$ there is some $b \in C$ which contains (with respect to E) exactly those elements which are ordered pairs (y, z), in C, such that a contains (with respect to E) the pair (z, y).

Closure under π_{23} and π_{123} are similar. Consider the latter. Let $a \in B$. Then there is some $b \in C$ such that for any $x \in C$ we have xEb iff there are $y, z, w \in C$ such that $x = (y, z, w)$ and $(z, w, y)Ea$. Now there is some $b' \in B$ such that for any $x \in B$ we have xEb iff there are $y, z, w \in B$ such that $x = (y, z, w)$ and $(z, w, y)Ea$.

Recall that we take power set closure to mean that for each $a \in C$ there is a $b \in C$ such that C satisfies $\forall z[zEb \leftrightarrow z \leq a]$. So if $a \in B$, there is $b' \in B$ such that for all $z \in B$, we have $zEb' \leftrightarrow z \leq a$, i.e. B satisfies $\forall z[zEb' \leftrightarrow z \leq a]$. Thus B is closed under power set formation in the sense we have been using. Since B is extensional, for $a \backslash z \neq 0$, we have $a \backslash z$ non-empty in B. So an extensional S-subalgebra B is closed under power sets in the stronger sense that $\forall a \exists b \forall z[zEb \leftrightarrow \forall x[xEz \rightarrow xEb]]$.

Preservation of closure under the component projections follows from the definitions

$$\text{(3-19)} \qquad \pi_1((y,z)) = \bigwedge\left(\bigwedge(y,z)\right) \qquad \pi_2((y,z)) = \bigwedge\left(\bigvee(y,z) \backslash \bigwedge(y,z)\right)$$

In these definitions the unary union and intersection apply only to finite sets. So closure under the projections follows from two facts. First, C is closed under binary union and intersections. Second, when the unary operators apply to singletons,

extensionality implies that the singleton's element is in the subalgebra. So we have $\bigwedge S(y) = \bigvee S(y) = y$ as required for the definitions above.

Preservation of elements which are (codes for) the projections follows from the observation of the previous paragraph. These projections only contain ordered pairs. As we construct an S-subalgebra from a sparse quotient, we easily recognize whether each pair belongs in π_j or not. ∎

CHAPTER 4

A MODEL OF STRICTLY QUANTIFIER-FREE COMPREHENSION

4.1. Preliminary Discussion

In this chapter we use a direct limit construction to give an extensional structure C which satisfies closure under a set of operations similar to Gödel operations. This closure will imply that it is a model of strictly quantifier-free comprehension with arbitrary parameters p. The structure will also be closed under power set formation, and it will contain (a code for) the standard natural numbers ω and a one-to-one embedding of u into ω. By Theorems 3.8 and 3.9, that model has an extensional substructure which is also closed under the Gödel-like operations, and hence also satisfies strictly quantifier-free comprehension with arbitrary parameters. The comprehensive extensional substructure will also contain a code for ω and a one-to-one embedding of the universe into ω. We will construct each stage of the limit system inside a fixed countable subalgebra A of $\langle {}^{\mathbf{N}}2, \vee, \wedge, \sim, \mathbf{0}, \mathbf{1}\rangle$. To define membership we expand the Boolean structure to include a singleton function S. The values of S will be atoms of ${}^{\mathbf{N}}2$, i.e. characteristic functions $\chi_{\{j\}}$ of actual singletons in ω.

As a preliminary step in the construction, we will establish a countable environment to work in, since there are only countably many atoms available in ${}^{\mathbf{N}}2$ as possible values for the one-to-one singleton function. Choose a countably infinite set G of free generators g such that any Boolean combination of $g_1, \ldots, g_k \in G$ other than $\mathbf{0}$ or $\mathbf{1}$ has support which is infinite and co-infinite in ω. The algebra generated by these elements is a sparse algebra where the Boolean combinations of the generators overlap arbitrarily, as in a generic Venn diagram, with enough space in each area for

infinitely many elements. Some spaces will turn out to be extensionally empty. This is rectified when we pass to an extensional substructure. The countable environment A is the Boolean subalgebra of ${}^{\mathbf{N}}2$ generated by G with the countably many atoms of ${}^{\mathbf{N}}2$. Enumerate A without repetition as $A = \{a_j\}_{j\in\omega}$. To distribute singletons of objects x, we will use an index function ind : $A \xrightarrow{1-1} \omega$ defined by $a_{\mathrm{ind}(x)} = x$.

The initial structure C_0 will be an S-algebra in ${}^{\mathbf{N}}2$ isomorphic to the minimal S-algebra. In an intermediate structure C_{n+1}, we need to make sure to use only finitely many generators which correspond to terms which represent Gödel-like operations on objects from previous structures. The membership relation of the previous structures will tell us all we need to know to correctly place the singletons of objects in C_{n+1}. The infinite support of the generators will allow us to map any object x at stage C_n to $\varphi_n(x)$ at the next stage C_{n+1}. We need to use only finitely many of the generators at a time, since the intersection of infinitely many of them might be too small to dominate infinitely many distinct atoms.

The major technical difficulty will be to define the connecting homomorphisms and verify that they preserve the singleton function. In C_{n+1} we will accomodate new generators which were ignored in the construction of C_n. Thus, in all probability, these maps are not simply inclusions. However, the generators are "free" because the elements of the Boolean algebra they generate are infinitely distant from each other. Specifically, at each stage no Boolean relation holds between any generators, so we are not constrained when we map them into the next stage. The generators will in fact remain fixed by the connecting homomorphisms of the limit system. As more free generators are added, the singleton function defined at that stage becomes more constrained. Within the limits imposed by the old generators, the singleton of an object is redefined or adjusted to make the associated membership relation correct with respect to the intended role of the new generators. This procedure of

restarting the construction at each stage from a sparse algebra (one where all elements are infinitely distant from each other) is also used to maintain closure under the pairing function and ensure that the ordered pairs they code are treated in a manner consistent with the treatment of the pairs' components. The connecting morphisms will be one-to-one.

The general strategy is to take a Boolean algebra with singleton function and, to each structure C_{n+1} in ${}^{\mathbf{N}}2$, add finitely many free generators g_τ which correspond to some Gödel-like terms τ applied to members of the previous structures. We will include single generators for the distinguished elements ω and f, the embedding of u into ω. The map φ_n is the identity on generators, and it is one-to-one on all of C_n. We adjust the singleton function at each stage C_n so that it makes the new generators g_τ act as τ interpreted in C_{n+1}, when we eventually define that structure. To accomplish this and maintain the roles of the previous generators, we use the fact that the intersection of any Boolean combination of generators has infinite support. It is important to keep in mind that, as we define each C_n (and φ_n), we *always* return to a sparse algebra $C_{n,0}$ and define C_n (and φ_n) by induction on m in $C_n = \bigcup\{C_{n,m}\}_m$. Then we can distribute singletons of objects in that space in accordance with the objects' indices. This use of the indices will ensure that the singleton function is one-to-one. Again, when we have a non-extensional subalgebra with a singleton function and an element of infinite support and finite extension, we can subtract finitely many singletons from it to produce an "atom" relative to the subalgebra. We will only use atoms of ${}^{\mathbf{N}}2$ as singletons and generally not refer to other objects as atoms.

The objects ω and $f : u \xrightarrow{1-1} \omega$ will be treated much as in Chapter 2. Our claims about them are structural. Their trivial "definitions" will be just vEc_ω and vEc_f. These formulas carry no information about these objects. We will verify our claims

about these objects by observation of the construction itself, rather than any general result. We associate these objects with formulas since that is the mechanism we will use to ensure closure under the Gödel-like operations.

By Propn 3.5, every $x \in C_n$ has a unique normal representation as

$$x = (q \setminus \bigvee \mathrm{S}_n[Y]) \vee \bigvee \mathrm{S}_n[Z] \tag{4-1}$$

where q is a Boolean combination of generators g_τ from C_n and $Y, Z \subset C_n$ are finite, with $\bigvee \mathrm{S}_n[Y] < q$ and $q \wedge \bigvee \mathrm{S}_n[Z] = \mathbf{0}$. This normal form is a consequence of the fact that the Boolean combinations of those generators for C_n are infinitely distant from each other. To prove φ_n is one-to-one, we will use an induction on the complexity of this normal representation of elements.

We will sometimes use objects in formulas. For example, we satisfy the comprehension axiom for $c_p Ev$, or simply pEv, with the principal ultrafilter for p, which we denote $\mathrm{puf}(p)$. We will need to denote every object x in our structure by a constant c_x. Our construction will contain objects e, i, ω and f as described above, and we will suppress the distinction between these objects and the corresponding constants.

Now, we begin rigorous development of the direct limit construction. For this construction, more complex than Chapter 2, we will expand the notion of G-term τ with special constants for a given structure, and we expand the operator $\mathrm{Def}[\tau]$.

DEFN 4.1. If C is any Boolean algebra, a *G-term with special constants for* C is any closed term τ in the language of e, i, ω, f, S, $\wedge$, $\vee$, $\sim$, $\times$, inv, π_{23}, π_{123}, $\mathcal{P}$, puf and $\{c_x | \, x \in C\}$, where $\{c_x | \, x \in C\}$ is a set of *special constant* symbols for every $x \in C$. In a *standard* G-term τ, special constants c_x and function symbols S, puf and $\mathcal{P}$ can only occur in subterms $\mathrm{S}(c_x)$, $\mathrm{puf}(c_x)$ and $\mathcal{P}(c_x)$. Define $T(C)$ to be the set of standard G-terms with special constants for C. Now we recursively define an operator Def

which applies to some formal terms $\tau \in T(C)$ for a membership structure $\langle C, E\rangle$. It yields a definition for the term via an open formula of $\mathcal{L}(E, \leq, =)$ with special constants for C.

(4-2)
$$\begin{aligned}
\mathrm{Def}[e](v) &\text{ is } \exists v_1 \exists v_2\, [v = (v_1, v_2) \wedge v_1 E v_2] \\
\mathrm{Def}[i](v) &\text{ is } \exists v_1 \exists v_2\, [v = (v_1, v_2) \wedge v_1 = v_2] \\
\mathrm{Def}[\omega](v) &\text{ is } [v E c_\omega] \\
\mathrm{Def}[f](v) &\text{ is } [v E c_f] \\
\mathrm{Def}[\sigma_1 \wedge \sigma_2](v) &\text{ is } [\mathrm{Def}[\sigma_1](v) \wedge \mathrm{Def}[\sigma_2](v)] \\
\mathrm{Def}[\sigma_1 \vee \sigma_2](v) &\text{ is } [\mathrm{Def}[\sigma_1](v) \vee \mathrm{Def}[\sigma_2](v)] \\
\mathrm{Def}[\sim \sigma](v) &\text{ is } [\sim \mathrm{Def}[\sigma](v)] \\
\mathrm{Def}[\sigma_1 \times \sigma_2](v) &\text{ is } \exists v_1 \exists v_2\, [v = (v_1, v_2) \wedge \mathrm{Def}[\sigma_1](v_1) \wedge \mathrm{Def}[\sigma_2](v_2)] \\
\mathrm{Def}[\mathrm{inv}(\sigma)](v) &\text{ is } \exists v_1 \exists v_2\, [v = (v_1, v_2) \wedge \mathrm{Def}[\sigma](v_2, v_1)] \\
\mathrm{Def}[\pi_{23}(\sigma)](v) &\text{ is } \exists v_1 \exists v_2 \exists v_3\, [v = (v_1, v_2, v_3) \wedge \mathrm{Def}[\sigma]((v_1, v_3, v_2))] \\
\mathrm{Def}[\pi_{123}\,\sigma)](v) &\text{ is } \exists v_1 \exists v_2 \exists v_3\, [v = (v_1, v_2, v_3) \wedge \mathrm{Def}[\sigma]((v_2, v_3, v_1))] \\
\mathrm{Def}[\mathrm{S}(c_p)](v) &\text{ is } [v = c_p] \\
\mathrm{Def}[\mathrm{puf}(c_p)](v) &\text{ is } [c_p E v] \\
\mathrm{Def}[\mathcal{P}(c_p)](v) &\text{ is } [v \leq p]
\end{aligned}$$

We will be concerned with closing S-algebras under the *special Gödel-like operations*, i.e. those other than Boolean operations and the singleton function, since we know every S-algebra is closed under the Boolean and singleton operations. An S-algebra C is *pointwise definable* if for every x in C there is a standard G-term τ such that we have $\langle C, E, \leq\, \rangle \vDash \forall v\, [v E x \leftrightarrow \mathrm{Def}[\tau](v)]$. (Outside of sub-terms $\mathrm{S}(c_p)$, $\mathrm{puf}(c_p)$ and $\mathcal{P}(c_p)$, we only allow the special constants for e, i, ω, and f to occur in our standard

G-terms.) We will show that an S-algebra C is closed under Gödel-like operations if it is pointwise definable and *has an interpretation for each standard G-term*, i.e. for each standard G-term τ there is some x in C such that $\langle C, E, \leq \rangle \models \forall v\,[vEx \leftrightarrow \mathrm{Def}[\tau](v)]$.

Terms $\mathrm{puf}(c_x)$ must refer to an object x in a structure rather than to a term τ, as in $\mathrm{puf}(\tau)$. In our non-extensional structures, the term τ may have multiple interpretions x and x' which both contain the same members but belong to different objects. Then it would not be clear which objects should be made elements of $\mathrm{puf}(\tau)$. In fact, this point arises at the end of the proof of Lemma 4.3. Similarly, we use special constants c_p in $\mathrm{Def}[\mathrm{S}(c_p)]$ since a term σ which defines p in a non-extensional structure will also define other objects' extensions, and we want the interpretation of $\mathrm{S}(c_p)$ to contain only one element. Also we use $\mathcal{P}(c_x)$ for a similar reason.

PROPN 4.2. *If an S-algebra C is pointwise definable and has an interpretation for each standard G-term with special constants for C, then C is closed under Gödel-like operations.*

PROOF: Since C has interpretations for G-terms e and i, it does contain codes for E and the equality relation. The structure has interpretions for ω and f, but these are irrelevant to this result, and they are unrestricted in this context. Every S-algebra C is closed under the Boolean operations and singletons.

C is closed under cartesian products. If $a, b \in C$ then, by pointwise definability, there are terms τ and σ which define a and b respectively, i.e. $\forall y[yEc \leftrightarrow \mathrm{Def}[\tau](y)]$ and $\forall z[zEb \leftrightarrow \mathrm{Def}[\sigma](z)]$. So C has an interpretation c for the standard G-term $\tau \times \sigma$. For every $x \in C$ we know $\langle C, E, \leq \rangle$ iff $\exists y, z[x = (y, z) \wedge \mathrm{Def}[\tau](y) \wedge \mathrm{Def}[\sigma](z)]]$. Thus, the element c is a cartesian product of a and b.

C is closed under inverses. If $a \in C$ then some standard G-term τ defines a, i.e. $\forall x[xEa \leftrightarrow \mathrm{Def}[\tau](x)]$. Then $\mathrm{inv}(a)$ is a standard G-term, and C has some inter-

pretation b for it. So $\langle C, E, \leq \rangle$ satisfies $\forall x[xEb \leftrightarrow \exists y, z[x = (z, y) \wedge \mathrm{Def}[\tau]((y, z))]$. The (unrestricted) special objects f, and also ω in this context, may contain ordered pairs. Suppose $\tau = \mathrm{inv}(c_f)$. Then C has an interpretation a of τ, i.e. $\langle C, E, \leq, c_f \rangle$ satisfies $\forall x[xEa \leftrightarrow \exists y, z[x = (z, y) \wedge (y, z)Ec_f]]$. Thus our trivial definitions of ω and f are sufficient to ensure closure under the essential operations. The argument permutations are similar to inverses.

For any $p \in C$, the term $\mathrm{puf}(c_p)$ is a standard G-term for C. So C has an interpretation y for it. Then $\langle C, E, \leq \rangle \vDash \forall x[xEy \leftrightarrow pEx]$. Thus C is closed under formation of principal ultra-filters as well

For any $p \in C$, the term $\mathcal{P}(c_p)$ is a standard G-term for C. So C has an interpretation y for it. Then $\langle C, E, \leq \rangle \vDash \forall x[xEy \leftrightarrow p \leq x]$. Thus C is closed under power set formation.

So C is closed under Gödel-like operations. ∎

We now present a technical lemma which we will use to verify properties of our direct limit system. This lemma again illustrates the significance of the rank structure. It says that so long as we generate S-algebras from a sparse Boolean subalgebra, satisfaction of $\mathrm{Def}[\tau]$ is preserved when we pass between an S-algebra and its various ranks, and when we pass between ranks of one S-algebra and another generated by a larger Boolean subalgebra. The reason is that $\mathrm{Def}[\tau](x)$ typically just says two simple things about x. One is that x has an associated finite binary tree structure as an ordered pair of ordered pairs of components of lower rank. The other part of $\mathrm{Def}[\tau]$ is that a certain quantifier-free formula holds among the components. Generation from a sparse Boolean subalgebra ensures that the smaller algebra never fails to recognize the existence of such an ordered pair decomposition. Of course preservation of the quantifier-free part in a direct limit is not problematic.

In our typical applications, the algebras are inside ${}^{\mathbf{N}}2$ and the non-zero elements of Q have infinite support. In an arbitrary S-algebra, this condition is somewhat different, since we cannot directly refer to the support in an algebra which corresponds to ${}^{\mathbf{N}}2$. In such a situation, the relevant consequence of sparseness is that Q contains no ordered pairs. So we can formulate this lemma in terms of the condition that for all $x_1, \ldots, x_k \in C$, we have $\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k) \notin Q$. Any such Q contains no ordered pairs. Our direct limit construction, later in this chapter, requires generators that dominate infinitely many atoms (relative to a larger algebra). The reason is only so that, in the event that these generators turn out to have infinite extension, they will have room for infinitely many singletons under them.

LEMMA 4.3. *Suppose* $C = \mathrm{SA}(Q)$ *where* Q *is a Boolean subalgebra of* C *and for all* $x_1, \ldots, x_k \in C$, *we have* $\mathrm{S}(x_1) \vee \cdots \vee \mathrm{S}(x_k) \notin Q$. *Suppose that* Q' *is a Boolean subalgebra of* Q. *Suppose we have two rank indices* $\alpha \leq \beta \leq \omega$, *and these refer to ranks with respect to the same singleton function* S. *Let* C'_α *be the union of the first* α *ranks of* $\mathrm{SA}(Q')$ *with respect to* Q' *and let* C_β *be the union of the first* β *ranks of* $\mathrm{SA}(Q)$ *with respect to* Q. *Then for all* $x \in C'_\alpha$

$$\langle C'_\alpha, E\rangle \vDash \exists y, z\,[x = (y, z)] \Leftrightarrow \langle C_\beta, E\rangle \vDash \exists y, z\,[x = (y, z)] \tag{4-3}$$

Further suppose that $T' \subset T(C'_\alpha)$ *is closed under subterms. Then for all* $x \in C'_\alpha$ *and* $\tau \in T'$ *we have*

$$\langle C'_\alpha, E\rangle \vDash \mathrm{Def}[\tau](x) \Leftrightarrow \langle C_\beta, E\rangle \vDash \mathrm{Def}[\tau](x) \tag{4-4}$$

PROOF: The ($\Rightarrow$) direction of (4-3) is immediate from $C'_\alpha \subset C_\beta$. The converse direction is non-trivial, because $x = (y, z)$ is not quantifier-free when it is interpreted

in $\mathcal{L}(E,=)$. This converse follows from the fact that for all x,y and z in C'_α, we have $\langle C'_\alpha, E\rangle \models [x = (y,z)] \leftrightarrow \langle C_\beta, E\rangle \models [x = (y,z)]$, which we prove by an easy induction on rank $\rho(x) \leq \alpha$. If $\rho(x) = 0$, then x is not a sup of singletons, so it is not an ordered pair in either structure. For $\rho = m+1$, if $\langle C'_\alpha, E\rangle \models [x = (y,z)]$, then $y, z \in C'_\alpha$ have rank at most $m - 1 < \alpha \leq \beta$. So $y, z \in C_\beta$, and we have $\langle C_\beta, E\rangle \models [x = (y,z)]$.

To prove (4-4) we begin with the initial cases of an induction on the height of τ. Suppose τ is e. If $x \in C'_\alpha$, then from (4-3) we know there are $y, z \in C'_\alpha$ such that C'_α satisfies $x = (y,z) \wedge yEz$ iff there $y, z \in C_\beta$ such that C_β satisfies $x = (y,z) \wedge yEz$. The case of i is similar. If τ is c_ω or c_f, then satisfaction of xEc_ω or xEf is preserved from C'_α to C_β, because $C'_\alpha \leq C_\beta \leq C$ and the formulas in question are atomic when x is taken as a specific parameter from C'_α.

Terms $\tau = \mathrm{S}(c_p)$ are trivial, because their defining formulas are $x = c_p$, which are atomic.

The case of $\tau = \sigma_1 \times \sigma_2$ is the easiest of the cases where the height of τ is non-zero. Let $x \in C'_\alpha$. Then

(4-5) $\langle C'_\alpha, E, \leq\rangle \models \mathrm{Def}[\sigma_1 \times \sigma_2](x)$

$\Leftrightarrow x = (y,z)$ for some $y, z \in C'_\alpha$ by definition (4-2) of $\mathrm{Def}[\sigma_1 \times \sigma_2]$

and $\langle C'_\alpha, E, \leq\rangle \models \mathrm{Def}[\sigma_1](y)$

and $\langle C'_\alpha, E, \leq\rangle \models \mathrm{Def}[\sigma_2](z)$

$\Leftrightarrow x = (y,z)$ for some $y, z \in C_\beta$ by induction hypothesis

and $\langle C_\beta, E, \leq\rangle \models \mathrm{Def}[\sigma_1](y)$

and $\langle C_\beta, E, \leq\rangle \models \mathrm{Def}[\sigma_2](z)$

$\Leftrightarrow \langle C_\beta, E, \leq\rangle \models \mathrm{Def}[\sigma_1 \times \sigma_2](x)$ by definition (4-2) of $\mathrm{Def}[\sigma_1 \times \sigma_2]$

Suppose $\tau = \mathrm{inv}(\sigma)$. Then for any $x \in C_\beta$ we have

(4-6) $\langle C'_\alpha, E, \leq \rangle \models \mathrm{Def}[\mathrm{inv}(\sigma)](x)$

$\Leftrightarrow x = (y,z)$ for some $y, z \in C'_\alpha$ by definition (4-2) of $\mathrm{Def}[\mathrm{inv}(\sigma)]$
and $\langle C'_\alpha, E, \leq \rangle \models \mathrm{Def}[\sigma]((z,y))$

$\Leftrightarrow x = (y,z)$ for some $y, z \in C_\beta$ by induction hypothesis
and $\langle C_\beta, E, \leq \rangle \models \mathrm{Def}[\sigma]((z,y))$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \models \mathrm{Def}[\mathrm{inv}(\sigma)](x)$ by definition (4-2) of $\mathrm{Def}[\mathrm{inv}(\sigma)]$

Suppose $\tau = \pi_{23}(\sigma)$. (The case of $\tau = \pi_{123}(\sigma)$ is similar.) Then for any $x \in C'_\alpha$ we have

(4-7) $\langle C'_\alpha, E, \leq \rangle \models \mathrm{Def}[\pi_{23}(\sigma)](x)$

$\Leftrightarrow x = (y,z,w)$ for some $y, z, w \in C'_\alpha$ by definition (4-2) of $\mathrm{Def}[\pi_{23}(\sigma)]$
and $\langle C'_\alpha, E, \leq \rangle \models \mathrm{Def}[\sigma]((y,w,z))$

$\Leftrightarrow x = (y,z,w)$ for some $w, y, z \in C_\beta$ by induction hypothesis
and $\langle C_\beta, E, \leq \rangle \models \mathrm{Def}[\sigma]((y,w,z))$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \models \mathrm{Def}[\pi_{23}(\sigma)](x)$ by definition (4-2) of $\mathrm{Def}[\pi_{23}(\sigma)]$

Terms $\tau = \mathrm{S}(c_p)$ are similar.

Now suppose $\tau = \mathrm{puf}(c_p)$ for some $p \in C'_\alpha$. Then for any $x \in C'_\alpha$ we have

(4-8) $\langle C'_\alpha, E, \leq \rangle \models \mathrm{Def}[\mathrm{puf}(c_p)](x)$

$\Leftrightarrow \langle C'_\alpha, E, \leq \rangle \models c_p E x$ by definition (4-2) of $\mathrm{Def}[\mathrm{puf}(c_p)]$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \models c_p E x$ since $\langle C'_\alpha, E, \leq \rangle \leq \langle C_\beta, E, \leq \rangle$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \models \mathrm{Def}[\mathrm{puf}(c_p)](x)$ by definition (4-2) of $\mathrm{Def}[\mathrm{puf}(c_p)]$

Note that we must assume that the object p itself is in C'_α. If C'_α just has an element p' which contains the same C'_α-members as p, then, since C_β and C may be non-extensional, p' may be contained in different objects x and have different principle ultrafilters.

Finally, suppose τ is $\mathcal{P}(c_p)$ for some $p \in C'_\alpha$. Then for any $z \in C'_\alpha$, we have

(4-9) $\langle C'_\alpha, E, \leq \rangle \vDash \mathrm{Def}[\mathcal{P}(c_p)](x)$

$\Leftrightarrow \langle C'_\alpha, E, \leq \rangle \vDash x \leq p$ by definition (4-2) of $\mathrm{Def}[\mathcal{P}(c_p)]$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \vDash x \leq p$ since $\langle C'_\alpha, E, \leq \rangle \leq \langle C_\beta, E, \leq \rangle$

$\Leftrightarrow \langle C_\beta, E, \leq \rangle \vDash \mathrm{Def}[\mathcal{P}(c_p)](x)$ by definition (4-2) of $\mathrm{Def}[\mathcal{P}(c_p)]$

Again, note that we must assume that the object p itself is in C'_α. Otherwise, it would not be clear which objects are supposed to be in $\mathcal{P}(c_p)$. ∎

In a direct limit system of S-algebras, if the images of an object all satisfy $\mathrm{Def}[\tau]$, then the next theorem will say that the object's image in the limit also satisfies $\mathrm{Def}[\tau]$. Thus, we can achieve closure under Gödel-like operations by stages. We will form terms with special constants for S-algebras C_j for $j \leq n$ to form the beginning of a direct limit system whose connecting morphisms are not inclusions. So different structures may conceivably contain an object which occurs in a different role in each structure. To be clear about which structure we want to use to interpret a term, we will use different constants for each structure C_n, and only use terms whose constants all refer to the same structure C_n.

DEFN 4.4. If $\{C_n\}_n$ and $\{\psi_n\}_n$ form a direct limit system of S-algebras, define $\psi^l_k : C_k \longrightarrow C_l$ to be the composition of connecting morphisms $\psi_{l-1} \circ \cdots \circ \psi_k$. If τ has special constants for C_k where $k < l$, then the *update of τ for C_l*, denoted τ^l, is

the term τ with each special constant c_p for C_k replaced by c_q where $q = \psi_k^l(p) \in C_l$. We say x *occurs first in* C_k if $k = 0$ and $x \in C_0$ or $x \in C_k$ and for all $0 \leq l < k$, we have $x \notin \psi_l^k[C_l]$. For $0 \leq l < k$ and any $x \in C_k$, we say x *originates in* C_l *as* x' iff $x = \psi_l^k(x')$ and x' occurs first in C_l.

If a standard G-term τ has special constants only for C_k, then τ and each of its updates σ will have associated generators g_τ and g_σ, respectively, in some later stage. They are intended to play the same role and they will be co-extensional in the limit. Of course we could eliminate some of this obvious redundancy. But we know of no way to eliminate all redundancy, which would make the limit structure extensional. So we will simply acknowledge this obvious redundancy. We do not try to eliminate it.

We must be somewhat careful that later structures still allow the generator associated with τ itself to play its intended role. This will be the case in our direct limit system by Lemma 4.3 and the fact that our connecting morphisms will be one-to-one. So we essentially have a union of a chain of structures. With terms which involve a special constant c_x for some C_k, if we let $y = \psi_l^k(x)$, then we will have a different constant c_y for C_l. From the definition $y = \psi_l^k(x)$ it is immediate that ψ is a homomorphism of the special constants, which are an implicit part of the language. Thus our use of special constants together with the simplicity of the Gödel-like operations ensures that we can make each new generator's updates continue to play the intended role even though ψ_n moves singletons and, at first, would seem to threaten changes in the membership relation from one stage to the next.

THEOREM 4.5. *Suppose* $C = \varinjlim\{C_n\}_n$ *is a direct limit of S-algebras with connecting morphisms* $\psi_l^k : C_k \xrightarrow{1-1} C_l$ *for* $l < k$. *Assume*

(i) *For each G-term* τ *with special constants for* C_j, *there is* $l > j$ *such*

that g_τ plays the role appropriate for τ, i.e. C_l satisfies the formula $\forall x[x \in g_\tau \leftrightarrow \mathrm{Def}[\tau](x)]$.

(ii) *For each τ and C_l as in (i), for all $k > l$, the element $\psi_l^k(g_\tau)$ plays the role in C_k appropriate for τ^k, i.e. C_k satisfies the formula*

$$\forall x[xE\psi_l^k(g_{\tau^k}) \leftrightarrow \mathrm{Def}[\tau^k](x)] \tag{4-10}$$

(iii) *Each C_l is pointwise definable.*

Then C is closed under Gödel-like operations, and C satisfies strictly quantifier-free comprehension with arbitrary parameters.

PROOF: The maps are one-to-one homomorphisms of $\wedge$, $\vee$, $\sim$ and S, and by definition of interpretations of the special constants, we have homomorphism of the special constants. Thus the maps preserve satisfaction of negative as well as positive sentences in $\mathcal{L}(\wedge, \vee, \sim, \mathrm{S}, =, c_x)$. So they also preserve all quantifier-free sentences in $\mathcal{L}(\wedge, \vee, \sim, \mathrm{S}, =, c_x)$. The direct limit preserves $\forall\exists$ sentences in $\mathcal{L}(\wedge, \vee, \sim, \mathrm{S}, =, c_x)$. The formulas in (4-10) are $\forall\exists$ in $\mathcal{L}(\wedge, \vee, \sim, \mathrm{S}, =, c_x)$, since $\mathrm{Def}[\tau]$ is existential and we interpret $x = (y, z)$ as $x = \mathrm{S}(\mathrm{S}(y)) \vee \mathrm{S}(\mathrm{S}(y) \vee \mathrm{S}(z))$, and we can interpret yEz as $\mathrm{S}(y) \leq z$, i.e. $(\mathrm{S}(y) \wedge z) = \mathrm{S}(y)$. Thus, satisfaction of (4-10) is preserved in the limit.

The terms e and i are interpreted in some C_l and are defined by existential formulas. Therefore, the corresponding elements of the direct limit also satisfy the appropriate definitions. Hence, C has codes for E and $=$. Again, the interpretations of ω and f are unrestricted in this context, since they are irrelevant to strictly quantifier-free comprehension.

For closure under cartesian products, suppose $a, b \in C$. Some C_j contains pre-images of both of them. Those pre-images are defined by standard G-terms τ and σ respectively, since each C_j is pointwise definable. It follows that $\mathrm{Def}[\tau \times \sigma]$ is a

standard G-term and has an interpretation x in some C_l. The images $\psi_l^k(x)$ interpret $\mathrm{Def}[\tau^k \times \sigma^k]$ in every C_k for $k > l$, and its image in C interprets the updated version of the term $\mathrm{Def}[\tau \times \sigma]$ in C. Thus, C is closed under cartesian products.

For closure under inverses, suppose $a \in C$. Some C_j contains a pre-image of it, and the pre-image is defined by a standard G-term τ. The term $\mathrm{inv}(\tau)$ has an interpretation x in some C_l. The images $\psi_l^k(x)$ interpret $\mathrm{Def}[\mathrm{inv}(\tau^k)]$ in every C_k for $k > l$, so its image in C interprets the updated version of the term $\mathrm{Def}[\tau]$ in C. Thus, C is closed under inverses. The other Gödel-like operations are similar, though puf, S and $\mathcal{P}$ use special constants for C, so we will consider the latter cases separately.

For any $p \in C$ with a pre-image x in some C_j, the terms $\mathrm{puf}(c_x)$, $\mathrm{S}(c_x)$ and $\mathcal{P}(c_x)$ are standard G-terms with special constants for C_j. So some C_l has an interpretation for each of the standard G-terms $\mathrm{puf}(c_y)$, $\mathrm{S}(c_y)$ and $\mathcal{P}(c_y)$, where y is the image of x in C_l. Each of those interpretations has an image z in C. The element z interprets $\mathrm{puf}(c_x)$, $\mathrm{S}(c_x)$ or $\mathcal{P}(c_x)$, respectively, since the defining formulas for these terms are atomic.

Thus, the limit C has interpretations for each of the standard G-terms. So by Propn. 4.5, the S-algebra C is closed under Gödel-like operations. Then by Theorem 2.23 we know that C satisfies strictly quantifier-free comprehension with arbitrary parameters and is closed under power set formation. ∎

In the situation described in the theorem, the image in the limit of an object which interprets a term τ will play the role indicated by τ. Then if we essentially close under terms of height n at stage $n+1$, we will have a limit which is closed under Gödel-like operations. Actually, we must add only finitely many generators at each stage. There are infinitely many terms of height less than or equal to n with special constants for stages up to stage n. So we will distribute the additional objects which

interpret those terms throughout the stages after stage n. Thus the limit will have an interpretation for every term τ.

4.2. The Direct Limit Construction

This section is an extended definition of an S-algebra by a direct limit construction in ${}^{\mathbf{N}}2$. The limit will be closed under Gödel-like operations. It will satisfy strictly quantifier-free comprehension and contain an element which embeds its universal set into its copy of ω.

We will generate each stage C_n rank by rank, much like $\mathrm{SA}(\mathbf{0},\mathbf{1})$. It follows that $C_n = \cup\{C_{n,m}\}_m$. Given C_n we will choose finitely many standard G-terms with special constants for C_n to be interpreted by elements of C_{n+1}. To do this we choose finitely many free generators in $A \subset {}^{\mathbf{N}}2$ whose non-zero Boolean combinations have infinite support, and thus form a sparse subalgebra $C_{n+1,0}$ of ${}^{\mathbf{N}}2$. The supports of the minimal non-zero elements (atoms relative to $C_{n+1,0}$) form a finite partition of ω. This partition refines the similar partition from $C_{n,0}$. To accomodate the new generators in $C_{n+1,0}$, we use information about the structure of C_n to adjust $\mathrm{S}_n(x) \mapsto \varphi_n(\mathrm{S}_n(x))$ within its partition set from C_n. Then we can generate C_{n+1} by ranks. So $C_{n+1} = \cup\{C_{n+1,m}\}_m$, since for any $x_1,\ldots,x_k \in C_{n+1,m}$, we have $\mathrm{S}_{n+1,m}(x_1) \vee \cdots \vee \mathrm{S}_{n+1,m}(x_k) \notin C_{n+1,0}$. Moreover, since $C_{n+1,0}$ is sparse in ${}^{\mathbf{N}}2$, we can easily scatter singletons of new elements within their appropriate partition sets to ensure that $\mathrm{S}_{n+1,m}$ is one-to-one.

This construction reveals a critical feature of our comprehension scheme. Circularity of definition is rather narrowly avoided for strictly quantifier-free comprehension, as one might expect. This is accomplished by extensive use of induction, especially in closure under $\mathrm{puf}(p)$ to allow parameters in the comprehension scheme. We have already seen that the witness of the comprehension axiom for xEe has mem-

bers $x = (y, z)$ determined by reference to satisfaction of xEy, not quite circular. The witness of the comprehension axiom for pEv has members determined by reference to satisfaction of vEp, again, not quite circular. Of course, an inductive construction cannot have the information required at each stage to determine membership in witnesses of the comprehension axioms for xEx or for $\exists w[w = v \wedge wE\!\!\!/v]$, since both formulas define Russell's set. These are formulas where the number of variables is reduced either by explicit quantification or by variable repetition, which in this instance has the same effect. Hence we claim that the problem with the paradoxes of Russell and Cantor is associated with binding variables, not with set theoretic interpretation of the basic aspects of logic.

Index the infinite set $G \subseteq A$ of free generators so $G = \{g_\tau | \tau \in T(A)\}$. We will essentially arrange that $\tau^C = g_\tau$. Technically, if g_τ originates in C_j, then τ^C will be a constant sequence $\langle g_\tau \rangle_{j \geq n} \in \varinjlim C_n$. For $\tau = \mathfrak{c}_\omega$, the E-elements of g_τ with the relation E will form a structure isomorphic to ω with $\in$.

To construct C_n, we will construct Boolean algebras $\{C_{n,m}\}_m$, each with a function $\mathrm{S}_{n,m}$ which defines singletons for elements of that structure. For many $x \in C_{n,m}$ the singleton $\mathrm{S}_{n,m}(x)$ will not be an element of $C_{n,m}$. The second index m is associated with iterated extension of $C_{n,m}$ by new atoms which are singletons of its elements, with consequent closure of C_n under S, as well as $\wedge$, $\vee$, and $\sim$. The first index n is associated with iterated addition of finitely many new generators at each step to achieve closure under our Gödel-like operations, which include standard terms of arbitrary finite height in $\mathcal{L}(e, i, \times, \mathrm{inv}, \pi_{23}, \pi_{123}, \mathrm{puf}, \mathcal{P})$.

The ranking of elements by the second index m is by far the more important rank. It relates to the repeated returns we make to sparse algebras in order to make S_n well-defined and to make φ_n preserve singletons. In addition, the m-ranks of the direct limit structure allow us to pass *downward* to an extensional substructure

of the limit C which preserves satisfaction of essentially Σ_1 formulas and others of even higher quantifier complexity. The formulas in question are relatively simple Σ_1 formulas. Their preservation does not imply the preservation of all Σ_1 formulas betweeen these structures. However, in $\mathcal{L}(E, \leq, =)$ we do need more than just downward preservation of quantifier-free and $\forall$ formulas.

So the Boolean algebra $C_{n,m}$ will not be closed under singletons. The S-algebra generated by S_n from $C_{n,0}$ will be C_n, where S_n is the common extension of $\{\mathrm{S}_{n,m}\}_m$, which we will define. Specifically, the Boolean subalgebra $C_{n,m+1}$ will be the Boolean extension of $C_{n,m}$ by the singletons of elements in that structure. In other words $C_{n,m+1} = C_{n,m}(S_{n,m}[C_{n,m}])$. The functions $\{\mathrm{S}_{n,m}\}_m$ will be compatible because of the facts that $C_{n,0}$ is sparse and that for any y, z we code (y, z) so it has finite support. Consequently, whenever an ordered pair is constructed, its components will already be present, and $\mathrm{S}_{n,m}$ will recognize it as an ordered pair.

Since we must have $\varphi_n(\mathrm{S}_n(x)) = \mathrm{S}_{n+1}(\varphi_n(x))$, our decision about each standard G-term τ and associated generator g_τ, whether $\varphi_n(\mathrm{S}_n(x)) \leq g_\tau$ or not, will determine whether (the image of) x remains a member of g_τ. It will also determine whether the image of x becomes a member of a new generator g_τ. Aside from $\tau = c_\omega, c_f$, for any other term τ in a specified finite $T_{n+1} \subset T$ and for any $x \in C_{n,m}$, we will indicate the "restriction on $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ due to the presence of τ in T_{n+1}" by

$$\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau)$$

This will be an element of ${}^{\mathbf{N}}2$, either g_τ or $\sim g_\tau$. It will indicate whether we need to place $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ under the generator g_τ or not.

For $\tau = \omega, f$, we have special cases because our claim about ω is structural; it is not definable by a first order formula. We will define $\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau)$ much as

we defined $R^S(x,\omega)$ and $R^S(x,f)$ in Chapter 2, since the function S_{n+1} will extend $\varphi_n[S_n]$. Now the j' of the construction in Chapter 2, a (code for a) natural number, may vary from one stage to another, so we will denote them $j^{(n)}$ for any $j, n \in \omega$. The only elements of ω in $C_{0,m+1}$ will be $0^{(0)} = \mathbf{0}$ and $(j+1)^{(0)} = j^{(0)} \vee S_{0,m}(j^{(0)})$ for $j = 0, \ldots, m$. The only elements of ω in $C_{n+1,m}$ will be $\varphi_n(j^{(n)})$ for $j \leq m$. We will make the natural identification of an element in the limit with an element in some stage C_n. For example, we identify $\langle f \rangle_{\alpha > n}$ for some n, with f itself. Since the connecting morphisms are one-to-one, the direct limit is essentially a union of a chain of S-algebras. This is why we can expect ω in the limit to be (isomorphic to) the standard natural numbers.

Recall that the only elements of f in Chapter 2 had rank at least two, and they were ordered pairs $x = (y,z)$, where $z = (\mathrm{ind}(y))'$, so $zE\omega$. Again, the element y may be moved by φ_n, with resultant changes in its index. So we need to require that $z = (\mathrm{ind}(w))^{(n)}$, where y originates as w in C_k, for some $k \leq n$. This definition in terms of the origin w of y will ensure that the image of y under the coded function f will be well-defined, even though the updates of y various in various stages are different and have different indices.

These restrictions $R^{\varphi}_{n,m+1}$ will adjust $\varphi_n[S_n]$ to accomodate the new generators of T_{n+1}. We then define S_{n+1} similarly, so it will extend $\varphi_n[S_n]$. We will indicate "the restriction on $S_{n+1,m}(x)$ due to the presence of τ in T_{n+1}" by

$$R^S_{n+1,m}(x,\tau)$$

This will indicate whether we need to place $S_{n+1,m}(x)$ under g_τ or not in order to make this generator contain the correct elements. In that case x will be any object in $C_{n+1,m}$, the next and larger stage of the construction, not just $\varphi_n[C_n]$.

Now we detail the construction. For every $n, m \in \omega$ and $y, z \in C_{n,m}$, we will define the usual ordered pair function restricted to any $C_{n,m}$ as $C_{n,m}$ constructed.

(4-11) $$(y,z)_{n,m} = \mathrm{S}_{n,m+1}(\mathrm{S}_{n,m}(y)) \vee \mathrm{S}_{n,m+1}(\mathrm{S}_{n,m}(y) \vee \mathrm{S}_{n,m}(z))$$

For all $n \in \omega$ and $y, z \in C_n$, we will define $(y,z)_n$ similarly. Define the sparse algebra $C_{0,0} = \{\mathbf{0}, \mathbf{1}\}$. For all $m \in \omega$ define $\mathrm{S}_{0,m}(x) = \chi_{\{\mathrm{ind}(x)\}}$. Then the definition $C_{0,m+1} = C_{0,m}(\mathrm{S}_{0,m}[C_{n,m}])$ gives us S_0: $C_0 \xrightarrow{1-1} C_0$ and $C_0 = \mathrm{SA}(\mathbf{0}, \mathbf{1})$, the smallest S-algebra in $\aleph 2$ which contains $\{\mathbf{0}, \mathbf{1}\}$. Let $T = T(A)$ be the set of standard G-terms with special constants for our chosen countable environment A. Technically, since we want different constants for each stage C_n, we will use $\aleph_0$ copies of each of these, with each copy distinguished by a stage index n. We will suppress this additional index. We will define C_{n+1} inductively via $T_0 = \emptyset$ and T_{n+1}; the finite set T_{n+1} is the set of standard G-terms in T which have height less than n and which have special constants c_x for some C_j with $j \leq n$, with $\mathrm{ind}(x) \leq n$. So T_{n+1} is finite and closed under subterms. We will define each $C_{n+1,0}$ to be the Boolean algebra generated by $\{g_\tau | \tau \in T_{n+1}\}$.

Assume for induction we have already constructed $C_0, \ldots, C_n$ and we have defined S-algebra homomorphisms $\varphi_0, \ldots, \varphi_{n-1}$ which are one-to-one. Let E_n be the associated membership relation for S_n.

To begin execution of the plan we have discussed to construct φ_n, let $\varphi_{n,0}$ be the identity on $C_{n,0}$. We want $\varphi_{n,m+1}$ to be a homomorphism of $\wedge$, $\vee$ and $\sim$ on $C_{n,m+1} = C_{n,m}(\mathrm{S}_{n,m}[C_{n,m}])$. Thus, to extend $\varphi_{n,m}$ to $C_{n,m+1}$, it suffices to define it on singletons, i.e. to define $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ for all $x \in C_{n,m}$. Define

(4-12) $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ is the $\mathrm{ind}(\varphi_{n,m}(x))^{\mathrm{th}}$ atom under

$$\bigwedge \{\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau) | \tau \in T_{n+1}\}$$

where $R^{\varphi}_{n,m+1}(S_{n,m}(x), \tau)$ for $\tau \neq \omega, f$ is defined by

(4-13) $$R^{\varphi}_{n,m+1}(S_{n,m}(x), \tau) = g_{\tau} \text{ iff } \langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\tau^n](x)$$

Recall that we refer to the associated membership relation E_n instead of S_n because C_n is not closed under S_n. The relation $\leq$ is simply the restriction of the order of ${}^{\mathbf{N}}2$ to C_n.

Consider the special case $\tau = \omega$. Technically, the maps φ_n are not inclusions, so we need to be careful. Each C_n will have different interpretations of the natural numbers. We will define

(4-14) $$0^{(n)} = 0 \text{ and } (j+1)^{(n)} = j^{(n)} \vee S_n(j^{(n)})$$

in C_n. Since $j^{(n)}$ is simply constructed from S-algebra operations, we see that we have $0^{(n)} \in C_{n,0}$ and $j^{(n)} \in C_{n,m}$ for each $j \leq m$. Thus we define

(4-15) $$R^{\varphi}_{n,m+1}(S_{n,m}(x), \omega) = g_{\omega} \text{ iff } x = j^{(n)} \text{ for some } j \leq m$$

We completee the definition of $R^{\varphi}_{n,m}$ with the other exceptional case where $\tau = f$. As described above, we define

(4-16) $$R^{\varphi}_{n,0}(x, f) = R^{\varphi}_{n,1}(x, f) = \sim g_f$$

and

(4-17) $R^{\varphi}_{n,m+2}(x, f) = g_f$ iff $\exists y, z \in C_{n,m}$ such that

$[x = (y, z)_{n,m},\ y$ originates as $w \in C_k,\ k < n$ and $z = (\mathrm{ind}(w))^{(n)}]$

To apply the definition of $\mathrm{R}^{\varphi}_{n,m}$, we start with $C_{n,0}$ on which $\varphi_{n,0}$ is defined to be the identity. We generate images for singletons of those objects and close under Boolean operations. That is to say, we generate images for elements of $C_{n,1}$. We repeat this in an induction on the second index m. When we define $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ for $x \in C_{n,m}$, we will already have defined $\varphi_{n,m}(x)$, so we will have its index available to use when we place the image of $\mathrm{S}_{n,m}(x)$.

With φ_n defined, to construct S_{n+1} and $C_{n+1} = \bigcup\{C_{n+1,m}\}_m$, we start with $C_{n+1,0} = \mathrm{BA}(g_\tau | \tau \in T_{n+1})$. A definition analogous to the definition (4-12,13) for $\varphi_{n,m+1}$ will suffice for $\mathrm{S}_{n,m+1}$. For $m = 0$, we have no $\mathrm{S}_{n+1,-1}$ available, so we cannot refer to any previous membership relation within stage $n+1$. However the situation is trivialized in many cases by our chosen codes for ordered pairs. In $C_{n+1,0}$ there are no ordered pairs since ordered pairs must be non-zero and have finite support. If τ is not $\mathrm{puf}(c_p)$, $\mathcal{P}(c_p)$, c_ω or $\mathrm{S}(c_p)$, then $\mathrm{Def}[\tau](x)$ (or the special definition of $\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,f)$, will require that x is an ordered pair. So

$$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,\tau) =\sim g_\tau \tag{4-18}$$

If $\tau = \mathrm{puf}(c_p)$ where c_p is a special constant for C_j for some $j \leq n$, let $p' = \varphi^n_j(p)$ to update τ. Then define

$$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,\tau) = g_\tau \text{ iff } \varphi_n(\mathrm{S}_n(p')) \leq x \tag{4-19}$$

Note that we do not know the rank of p' in C_n. So prior to construction of C_{n+1}, we must construct all of φ_n and of $\varphi_n[C_n]$, an isomorphic copy of C_n, which will be compatible with our intended use of the new generators in C_{n+1}. The atom $\varphi_n(\mathrm{S}_n(p'))$ gives us all the membership information about (the update of) p which we need at stage $n+1$.

If $\tau = \mathcal{P}(c_p)$ with $p \in C_j$ for some $j \leq n$ and $\mathrm{ind}(p) \leq n$, let $p' = \varphi_j^n(p)$ to calculate the update τ^n. Note that $\varphi_n(p')$ is φ_j^{n+1}. This is the appropriate update of p for the definition of restrictions $\mathrm{R}^{\mathrm{S}}_{n+1}$. Define

(4-20) $$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,\tau) = g_\tau \text{ iff } x \leq \varphi_n(p')$$

For $\tau = \omega$, since $C_{n+1,0}$ is sparse, we define

(4-21) $$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,\tau) = g_\tau \text{ iff } x = \mathbf{0}$$

Now we define

(4-22) $S_{n+1,0}(x)$ is the $\mathrm{ind}(x)^{\mathrm{th}}$ atom under $\bigwedge\{\mathrm{R}^{\mathrm{S}}_{n+1,0}(x,\tau) | \tau \in T_{n+1}\}$

These cases where $m = 0$ and $\tau = \mathrm{puf}(c_p)$, $\mathcal{P}(c_p)$, c_ω or $\mathrm{S}(c_p)$ are critical in the construction since they ensure that $\mathrm{S}_{n+1}(\varphi_n(x)) = \varphi_n(\mathrm{S}_n(x))$ in the fundamental instances. In the case of these non-pairs $x \in C_{n+1,0}$, for correct management of the singleton with respect to the fundamental comprehension axioms for the membership language, we are essentially just required to make φ_n be one-to-one and place$\varphi_n(\mathrm{S}_n(x))$ under u^0. Then as an ordered pair $x = (y,z)$ is constructed in C_{n+1}, our placement of $\mathrm{S}_{n+1}(x)$ is guided by what we have already done with the components of x. More non-pairs are created in $C_{n,m+1}$. They have the normal form

(4-23) $$x = \Big(q \setminus \bigvee \mathrm{S}_{n+1}[Y]\Big) \vee \bigvee \mathrm{S}_{n+1}[Z]$$

where $q \in C_{n+1,0}$, and $Y, Z \subset C_{n+1}$ are finite. Again, correct management of their singletons essentially just requires that φ_n is one-to-one and places them under u^0.

Now we detail the extension of $S_{n+1,0}$. Our definition of $xE_{n+1,m}y$ by the formula $S_{n+1,m}(x) \leq y$ will not require closure of $C_{n+1,m}$ under $S_{n+1,m}$, since the order referred to in the definition is the order of $\aleph_2$ and $S_{n+1,m}(x)$ is an element of $\aleph_2$.

To extend (4-22) for all m, we define $S_{n+1,m}$ as follows.

(4-24) $S_{n+1,m}(x)$ is the $\mathrm{ind}(x)^{\mathrm{th}}$ atom under $\bigwedge\{R^S_{n+1,m}(x,\tau) | \tau \in T_{n+1}\}$

where, for $m > 0$ and $\tau \neq \omega$, f

(4-25) $R^S_{n+1,m}(x,\tau) = g_\tau$ iff $\langle C_{n+1,m}, E_{n+1,m}, \leq \rangle \vDash \mathrm{Def}[\tau^n](x)$

For $\tau = \omega$, as with $R^{\varphi}_{n,m+1}$, we have a special case, because our claim about ω is not definable by a first order formula. Since $j^{(n+1)}$ is simply constructed from the S-algebra operations, we see that we have $0^{(n+1)} \in C_{n,0}$ and $j^{(n+1)} \in C_{n+1,m}$ for each $j \leq m$. In the definition of $j^{(n+1)}$, note that we only use the previously constructed part $S_{n,m}$ of the singleton function. Thus we define

(4-26) $R^S_{n+1,m}(x,\omega) = g_\omega$ iff $x = j^{(n+1)}$ for some $j \leq m$

For $\tau = f$, we define

(4-27) $R^S_{n+1,m}(x,f) = g_f \leftrightarrow \exists\, y, z \in C_{n,m-2}$

$[x = (y,z)_{n,m-2},\ y$ originates as w in $C_k,\ k \leq n+1$ and $z = (\mathrm{ind}(w))^{(n+1)}]$

Finally, we define $S_n = \cup\{S_{n,m}\}_m$ with $C_n = \cup\{C_{n,m}\}_m$ and $\varphi_n = \cup\{\varphi_{n,m}\}_m$. This finishes the construction of the direct limit system.

4.3. Strictly Quantifier-free Comprehension with Parameters

We proceed with our central technical result that $R^{\varphi}_{n,m+1}$ and $R^{S}_{n+1,m}$ are indeed compatible with respect to all $\tau \in T_{n+1}$. We have to verify three features of the construction of Section 4.2. One is that the maps φ_n are homomorphisms so that we have a direct limit system of S-algebras. Then, as we showed in Chapter 2, the limit C will be an S-algebra. Second, we need to verify the hypotheses of Lemma 4.3, so we know that C is closed under Gödel-like operations, essentially because the formulas which define those operations are $\forall\exists$ in $\mathcal{L}(\wedge, \vee, \sim, S)$. Finally, we need to observe that ω and f demonstrate internal countability of C.

We will distinguish those terms whose elements will necessarily be ordered pairs. Note that such "non-relational" terms may contain ordered pairs, but our analysis will not assume that they contain only ordered pairs.

DEFN. 4.6. A special G-term whose outermost symbol is ω, puf or $\mathcal{P}$ is a *non-relational* term. The other special G-terms are *relational*, i.e. those with outermost symbol e, i, f, $\times$, inv, π_{23} or π_{123}.

We will verify by induction on m that $\varphi_{n,m}$ is a homomorphism of $S_{n,m}$. Then it is a homomorphism of the ordered pair function. In all the cases where τ is relational, the situation is trivialized when x is not an ordered pair. Consider an ordered pair $x = (x_1, x_2)_{n+1,m-2}$. Since x is an ordered pair and τ is relational, we see that $R^{\varphi}_{n,m+1}$ and $R^{S}_{n+1,m}$ are compatible as follows. We have $x = (x_1, x_2)_{n+1,m-2}$ and $S_{n+1,m-2}(x_1)$ is already defined with $x_1 E x_2$ or $x_1 \not E x_2$ already decided. The important point here is not just that the definition is non-circular. At each stage of the induction on m, for each τ in T_{n+1}, our concern is whether g_τ serves as the interpretation of τ in C_{n+1}. The treatment by S_{n+1} of k-tuples is consistent with the treatment of the k-tuples' components, which arise in earlier ranks of C_{n+1},

associated with smaller values of m. This is also trivially true in C_0, since $T_0 = \emptyset$. The consequence of this is that φ_n will be a homomorphism of singleton functions so long as it preserves singletons of non-pairs. Nested within the induction on m, we will need induction on the complexity of τ and, in certain instances, also induction on the complexity of $x \in C_{n,0}$ as a Boolean combination of the generators g_σ. This will be tedious.

PROPN 4.7. *Let C_n be defined as in Section 4.2. For all $n \in \omega$, the map φ_n is a one-to-one homomorphism of the S-algebra C_n into C_{n+1}.*

PROOF: On $C_{n,0}$, we defined $\varphi_{n,0}$ to be the identity, which is one-to-one, and we used the Boolean normal form representation of $x \in C_n$ to extend φ_n, so φ_n is a Boolean homomorphism. Now it suffices to prove that $\varphi_n(\mathrm{S}_n(x)) = \mathrm{S}_{n+1}(\varphi_n(x))$. Compare their definitions, supposing that x has rank m in C_n.

(4-12) $\varphi_{n,m+1}(\mathrm{S}_{n,m}(x))$ is the $\mathrm{ind}(\varphi_{n,m}(x))^{\mathrm{th}}$ atom under

$$\bigwedge\{\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x),\tau) \mid \tau \in T_{n+1}\}$$

When we apply definition (4-24) of $\mathrm{S}_{n+1,m}$ to $\varphi_{n,m}(x)$, we have

(4-28) $\mathrm{S}_{n+1,m}(\varphi_{n,m}(x))$ is the $\mathrm{ind}(\varphi_{n,m}(x))^{\mathrm{th}}$ atom under

$$\bigwedge\{\mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x),\tau) \mid \tau \in T_{n+1}\}$$

Notice that to match these definitions required in part that we define $\varphi_n(\mathrm{S}_n(x))$ in terms of $\varphi_n(x)$. This self-reference in the definition of φ_n is the reason we need to define φ_n by induction over the rank index m.

To complete the match of (4-12) and (4-28), it will suffice to show

(4-29) $$\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x),\tau) = \mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x),\tau)$$

for all $\tau \in T_{n+1}$ and $x \in C_{n,m}$. We prove this by induction on m and on the complexities of τ and x. For each n, the initial case $m = 0$, the most complicated, is treated in the following lemma.

LEMMA 4.8. *For C as constructed in Section 4.2, for any $n \in \omega$, any $x \in C_{n,0}$ and all $\tau \in T_{n+1}$, we have*

$$\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = \mathrm{R}^{\mathrm{S}}_{n+1,0}(\varphi_{n,0}(x), \tau) \tag{4-30}$$

PROOF: First consider the easy cases of relational τ, where τ has only special constants for C_j for some $j < n$.

From definition (4-13) of $\mathrm{R}^{\varphi}_{n,m+1}$, we have

$$\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = g_\tau \text{ iff } \langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\tau^n](x) \tag{4-31}$$

Now since $x \in C_{n,0}$ is not an ordered pair, it follows that for relational τ we have $\langle C_n, E_n, \leq \rangle \nvDash \mathrm{Def}[\tau^n](x)$ and hence $\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = \sim g_\tau$.

For $\mathrm{R}^{\mathrm{S}}_{n+1,0}(x, \tau)$ with relational τ, from the definition (4-12) we have

$$\begin{aligned} \mathrm{R}^{\mathrm{S}}_{n+1,0}(\varphi_{n,0}(x), \tau) &= \mathrm{R}^{\mathrm{S}}_{n+1,0}(x, \tau) && \text{since } \varphi_{n,0}(x) = x \\ &= \sim g_\tau && \text{by definition (4-18) of } \mathrm{R}^{\mathrm{S}}_{n+1,0} \\ &= \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) && \text{as we just observed following (4-32)} \end{aligned} \tag{4-32}$$

The analysis of $\mathrm{R}^{\mathrm{S}}_{n+1,0}(x, \tau)$ is much more complicated in the critical cases where $m = 0$ and τ is non-relational, since the initial two ranks are non-trivial, and

also the rank of an object p referred to by a special constant c_p may be well above zero. Note again that $\varphi_{n,0}(x) = x$. Every $x \in C_{n,0}$ is a Boolean combination of generators $g_\sigma \in T_n$ where σ has special constants for C_j for some $j < n$. We will begin with consideration of the generators g_σ, and we consider the rest of $C_{n,0}$ by induction on the complexity of Boolean combinations of those generators g_σ. The most complicated case,with $m = 0$ and $\tau = \mathrm{puf}(c_p)$ is considered in the following lemma.

SUBLEMMA 4.8.1. *For C as constructed in Section 4.2, any $n \in \omega$ and $g_\sigma \in C_{n,0}$, if c_p is a special constant symbol for C_j for some $j \leq n$, then*

$$\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(g_\sigma), \mathrm{puf}(c_p)) = R^{\mathrm{S}}_{n+1,0}(\varphi_{n,0}(g_\sigma), \mathrm{puf}(c_p)) \tag{4-33}$$

PROOF: Consider $\tau = \mathrm{puf}(c_p)$, also denoted $\mathrm{puf}(p)$, for p as in the hypothesis. Let p' denote the update $\varphi^n_j(p)$ of p into C_n. The relevant instance of definition (4-13) of $\mathrm{R}^{\varphi}_{n,m+1}$ is

$$\begin{aligned}
(4\text{-}34)\ &\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathrm{puf}(p)) = g_{\mathrm{puf}(p)} \\
&\Leftrightarrow \langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\mathrm{puf}(p')](x) && \text{by definitions (4-13)} \\
&&& \text{of } \mathrm{R}^{\varphi}_{n,m+1} \text{ and (4.4) of } \tau^n \\
&\Leftrightarrow \langle C_n, E_n, \leq \rangle \vDash p' E x && \text{by definition (4-2) of } \mathrm{Def}[\mathrm{puf}(p')]
\end{aligned}$$

We have

$$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x, \mathrm{puf}(p)) = g_{\mathrm{puf}(p)} \leftrightarrow \varphi_n(\mathrm{S}_n(p')) \leq x = g_\sigma \tag{4-35}$$

by definition (4-19) of $\mathrm{R}^{\mathrm{S}}_{n+1,0}$ when $\tau = \mathrm{puf}(p)$. We want to relate these conditions to the restriction in (4-34) on $\varphi_n(\mathrm{S}_n(x))$. This is why we need to consider $x = g_\sigma$ and proceed from there by induction. We know

(4-36) $$\varphi_n(\mathrm{S}_n(p')) \leq x = g_\sigma \leftrightarrow \mathrm{R}^{\varphi}_{n,l+1}(\mathrm{S}_n(p'), \sigma) = g_\sigma$$

where $p' \in C_{n,l}$ and for all c_q in σ, the object q is in some $C_{k,l}$ with $k \leq n$. Now the right hand side is a condition on S_n instead of S_{n+1}. It is equivalent to

(4-37) $$\langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\sigma^n](p')$$

by definition (4-13) of $\mathrm{R}^{\varphi}_{n,l+1}$. That is equivalent to

(4-38) $$\langle C_{n,l}, E_{n,l}, \leq \rangle \vDash \mathrm{Def}[\sigma^n](p')$$

by Lemma 4.3, and hence equivalent to

(4-39) $$\mathrm{R}^{\mathrm{S}}_{n,l}(p', \sigma) = g_\sigma$$

since $\sigma \in T_n$. This is equivalent to

(4-40) $$\langle C_{n,l}, E_{n,l}, \leq \rangle \vDash p' E g_\sigma$$

by definition (4-24) of $\mathrm{S}_{n,l}$. This is equivalent to

(4-41) $$\langle C_n, E_n, \leq \rangle \vDash p' E g_\sigma$$

since $p' E g_\sigma$ is quantifier-free. Finally, this condition is equivalent to

$$\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathrm{puf}(p)) = g_{\mathrm{puf}(p)} \tag{4-42}$$

by (4-34).

This completes the proof of Sublemma 4.8.1. ∎

We will later use the observation that the cases $\tau = \omega, \mathcal{P}(c_p)$ will not depend on $x \in C_{n,0}$ being a generator.

Consider $\tau = \omega$. The relevant instance of definition (4-15) of $\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \omega)$ is $\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \omega) = g_\omega$ iff $x = 0^{(n)} = \mathbf{0}$. Since $\varphi_{n,0}(g_\omega) = g_\omega$, the definition (4-26) of $\mathrm{R}^{\mathrm{S}}_{n+1,m}$ tells us

$$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x, g_\omega) = g_\omega \text{ iff } x = 0^{(n+1)} = \mathbf{0} \tag{4-43}$$

Suppose $\tau = \mathcal{P}(c_p)$, where c_p is a special constant symbol for C_j for some $j \leq n$. Let p' denote the update $\varphi^n_j(p)$ of p into C_n. The relevant instance of definition (4-13) of $\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau)$ is

$$\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathcal{P}(p)) = g_{\mathcal{P}(p)} \text{ iff } x \leq \varphi_n(p') \tag{4-44}$$

Then

$$\begin{aligned}
&(4\text{-}45)\ \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathcal{P}(p)) = g_{\mathcal{P}(p)} \\
&\quad \leftrightarrow \langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\mathcal{P}(c_{p'})](x) \quad \text{by definitions (4-13)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{of } \mathrm{R}^{\varphi}_{n,m+1} \text{ and (4.4) of } \tau^n \\
&\quad \leftrightarrow x \leq p' \qquad\qquad\qquad\qquad \text{by definition (4-2) of } \mathrm{Def}[\mathcal{P}(c_{p'})]
\end{aligned}$$

By definition (4-20) of $\mathrm{R}^{\mathrm{S}}_{n+1,0}(y, \mathcal{P}(p))$ for $y = \varphi_{n,0}(x)$, we have

(4-46) $\mathrm{R}^{\mathrm{S}}_{n+1,0}(\varphi_n(x), \mathcal{P}(p)) = g_{\mathcal{P}(p)}$

$\Leftrightarrow \varphi_n(x) \leq \varphi_n(p')$ by definition (4-20) of $\mathrm{R}^{\mathrm{S}}_{n+1,0}$ when $\tau = \mathcal{P}(p)$

$\Leftrightarrow x \leq p'$ since φ_n is a Boolean homomorphism

$\Leftrightarrow \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathcal{P}(p)) = g_{\mathcal{P}(p)}$ by definition (4-13) of $\mathrm{R}^{\varphi}_{n,m+1}$ when $\tau = \mathcal{P}(p)$

Now Lemma 4.8 is proven for $x = g_\sigma \in C_{n,0}$. In the following sublemma, we finish Lemma 4.8 with a straightforward induction on normal form Boolean combinations of the generators.

SUBLEMMA 4.8.2. *For C as constructed in Section 4.2, for any $n \in \omega$, any $x \in C_{n,0}$ and any non-relational $\tau \in T_{n+1}$, we have* $\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = R^{\mathrm{S}}_{n+1,0}(\varphi_{n,0}(x), \tau)$.

PROOF: As noted above, the cases of Sublemma 4.8.1 with $\tau = \omega, \mathcal{P}(c_p)$ did not depend on the assumption that $x \in C_{n,0}$ was a generator, so those cases have already been considered.Thus we assume $\tau = \mathrm{puf}(p)$.

Suppose $x = \sim g_\sigma$ with $\sigma \in T_n$.

(4-47) $\mathrm{R}^{\mathrm{S}}_{n+1,0}(\sim g_\sigma, \mathrm{puf}(p)) = g_{\mathrm{puf}(p)}$

$\Leftrightarrow \varphi_n(\mathrm{S}_n(p')) \leq \sim g_\sigma$ by definition (4-19) of $\mathrm{R}^{\mathrm{S}}_{n+1,0}$

$\Leftrightarrow \varphi_n(\mathrm{S}_n(p')) \not\leq g_\sigma$ since $\varphi_n(\mathrm{S}_n(p'))$ is an atom

$\Leftrightarrow \langle C_n, E_n, \leq \rangle \vDash p' \mathrel{H} g_\sigma$ by definition (4-12,13) of $\varphi_n(\mathrm{S}_n(p'))$

$\Leftrightarrow \langle C_n, E_n, \leq \rangle \vDash p' E(\sim g_\sigma)$ since $\varphi_n(\mathrm{S}_n(p'))$ is an atom

$\Leftrightarrow \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(\sim \sigma), \mathrm{puf}(p)) = g_{\mathrm{puf}(p)}$ by (4-34)

For $x = x_1 \wedge x_2$, suppose that x_1 is either g_σ or $\sim g_\sigma$ for some $\sigma \in T_n$, and x_2 is also either g_σ or $\sim g_\sigma$ for some $\sigma \in T_n$.

(4-48) $\mathrm{R}^{\mathrm{S}}_{n+1,0}(x_1 \wedge x_2, \mathrm{puf}(p)) = g_{\mathrm{puf}(p)}$

$\Leftrightarrow \varphi_n(\mathrm{S}_n(p')) \leq x_1 \wedge x_2$ by definition (4-19) of $\mathrm{R}^{\mathrm{S}}_{n+1,0}$ with $\tau = \mathrm{puf}(p)$

$\Leftrightarrow \varphi_n(\mathrm{S}_n(p')) \leq x_1$ since $\varphi_n(\mathrm{S}_n(p'))$ is an atom and $\varphi_n(\mathrm{S}_n(p')) \leq x_2$

$\Leftrightarrow \langle C_n, E_n, \leq \rangle \vDash p' E(x_1 \wedge x_2)$ by definition (4-12,13) of $\varphi_n(\mathrm{S}_n(p'))$

$\Leftrightarrow \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \mathrm{puf}(p)) = g_{\mathrm{puf}(p)}$ by (4-34)

The case of $x = h_1 \vee h_2$ is similar. Thus, for all $x \in C_{n,0}$ and non-relational τ, we have

(4-49) $$\mathrm{R}^{\mathrm{S}}_{n+1,0}(x, \tau) = g_\tau \leftrightarrow \mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = g_\tau$$

This completes the proof of Sublemma 4.8.2. ∎

Now in our proof by induction over m to show that φ_n is a homomorphism on $C_{n,m}$, we have shown that $\mathrm{R}^{\varphi}_{n,1}(\mathrm{S}_{n,0}(x), \tau) = \mathrm{R}^{\mathrm{S}}_{n+1,0}(\varphi_{n,0}(x), \tau)$ for all $\tau \in T_{n+1}$, so $\varphi_{n,1}(\mathrm{S}_{n,0}(x)) = \mathrm{S}_{n+1,0}(\varphi_{n,0}(x))$. This ends the proof of Lemma 4.8. ∎

We also need to maintain in the induction hypothesis that φ_n is one-to-one on $C_{n,m}$. We consider only the initial case in detail.

LEMMA 4.9. *The map $\varphi_{n,1}$ is one-to-one.*

PROOF: This follows from the fact that $C_{n,0}$ is sparse. Let

$$x = (q \setminus \bigvee \mathrm{S}_{n,0}[Y]) \vee \bigvee \mathrm{S}_{n,0}[Z] \quad \text{and} \quad x' = (q' \setminus \bigvee \mathrm{S}_{n,0}[Y']) \vee \bigvee \mathrm{S}_{n,0}[Z'] \tag{4-50}$$

where we have $q \in C_{n,0}$ with $\bigvee \mathrm{S}_{n,0}[Y] \leq q$ and $q \wedge \bigvee \mathrm{S}_{n,0}[Z] = \mathbf{0}$, and similar conditions on x', q', Y' and Z'. Suppose $\varphi_{n,1}(x) = \varphi_{n,1}(x')$. If $q \neq q'$, then q' and q are infinitely distant. With $\varphi_{n,0}(q) = q$ and $\varphi_{n,0}(q') = q'$, it is impossible for

$$\varphi_{n,1}(x) = (q \setminus \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y])) \vee \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Z]) \tag{4-51}$$

to equal

$$\varphi_{n,1}(x') = (q' \setminus \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y'])) \vee \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Z']) \tag{4-52}$$

Thus, we have $q' = q$.

If $Z = \emptyset$, then

$$\varphi_{n,1}(x) = q \setminus \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y]) \tag{4-53}$$

equals

$$\varphi_{n,1}(x') = (q \setminus \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y'])) \vee \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Z']) \tag{4-54}$$

only if $Z' = \emptyset = Z$ and

$$\bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y']) = \bigvee \varphi_{n,1}(\mathrm{S}_{n,0}[Y]) \tag{4-55}$$

Then, since $S_{n,0}$ is one-to-one, we have $Y' = Y$.

Similarly, if $Z \neq \emptyset$, to have

$$(q \backslash \varphi_{n,1}(\bigvee S_{n,0}[Y])) \vee \varphi_{n,1}(\bigvee S_{n,0}[Z]) \tag{4-56}$$

equal

$$(q \backslash \varphi_{n,1}(\bigvee S_{n,0}[Y'])) \vee \varphi_{n,1}(\bigvee S_{n,0}[Z']) \tag{4-57}$$

requires $Z' = Z$ and $\bigvee S_{n,0}[Y'] = \bigvee S_{n,0}[Y]$, whence $Y' = Y$. Then we have $x' = x$. Thus $\varphi_{n,1}$ is one-to-one, and Lemma 4.9 is proven. ∎

To continue the induction on m, we need to show both that

$$R^{\varphi}_{n,m+1}(S_{n,m}(x), \tau) = R^{S}_{n+1,m}(\varphi_{n,m}(x), \tau) \tag{4-58}$$

for any $x \in C_{n,m}$ with $\tau \in T_{n+1}$, and that $\varphi_{n,m+1}$ is one-to-one. Our induction hypothesis for $\varphi_{n,m+1}$ is that

$$R^{\varphi}_{n,m}(S_{n,m-1}(x), \tau) = R^{S}_{n+1,m-1}(\varphi_{n,m-1}(x), \tau) \tag{4-59}$$

for all $x \in C_{n,m-1}$ with $\tau \in T_{n+1}$, and that $\varphi_{n,m}$ is one-to-one. Then it follows that $\varphi_{n,m}(S_{n,m-1}(x)) = S_{n+1,m-1}(\varphi_{n,m-1}(x))$. Since the definition of $(y,z)_{n,m-2}$ uses only $S_{n,m-1}$, for any $x \in C_{n,m}$ we have

$$\exists y, z \in C_{n,m-2}[x = (y,z)_{n,m-2}] \leftrightarrow \exists y', z' \in C_{n+1,m-2}[\varphi_{n,m}(x) = (y',z')_{n+1,m-2}] \tag{4-60}$$

in which case $y' = \varphi_{n,m-2}(y)$ and $z' = \varphi_{n,m-2}(z)$. Moreover, by definition of $E_{n,m}$, we have that $\varphi_{n,m}$ is a homomorphism of $E_{n,m}$ on $C_{n,m}$.

Suppose $\tau \neq \omega, f$. Similar to (4-31) and other parts of the case $m = 0$, we use definition (4-13) of $\mathrm{R}^{\varphi}_{n,m+1}$. For any x in $C_{n,m}$

(4-13) $$\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau) = g_\tau \text{ iff } \langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\tau^n](x)$$

In addition, if we let $C'_{n,m} = \varphi_{n,m}[C_{n,m}]$ and let $y = \varphi_{n,m}(x)$, then we have

(4-61) $$\begin{aligned} &\langle C_n, E_n, \leq \rangle \vDash \mathrm{Def}[\tau^n](x) \\ &\quad \leftrightarrow \langle C_{n,m}, E_{n,m}, \leq \rangle \vDash \mathrm{Def}[\tau^n](x) && \text{by Lemma 4.3} \\ &\quad \leftrightarrow \langle \varphi_{n,m}[C_{n,m}], E_{n+1,m}, \leq \rangle \vDash \mathrm{Def}[\tau^{n+1}](y) && \text{since } \varphi_{n,m} \text{ is an} \\ &\quad && E_{n,m}\text{-isomorphism} \\ &\quad \leftrightarrow \langle C_{n+1,m}, E_{n+1,m}, \leq \rangle \vDash \mathrm{Def}[\tau^{n+1}](y) && \text{by Lemma 4.3} \end{aligned}$$

Also, $\mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x), \tau) = g_\tau \leftrightarrow \langle C_{n+1,m}, E_{n+1,m} \rangle \vDash \mathrm{Def}[\tau^{n+1}](\varphi_{n,m}(x))$, by definition (4-25) of $\mathrm{R}^{\mathrm{S}}_{n+1,m}$.

Suppose $\tau = \omega$. Then $\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \omega) = g_\omega$ iff $x = j^{(n)}$ for some $j \leq m$. Also, $\mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x), \omega) = g_\omega$ iff $\varphi_{n,m}(x) = j^{(n+1)}$ for some $j \leq m$, which is equivalent to $x = j^{(n)}$ for some $j \leq m$, since $\varphi_{n,m}(\mathrm{S}_{n,m-1}(z)$ is $\mathrm{S}_{n+1,m-1}(\varphi_{n,m-1}(z))$ for any $z \in C_{n,m-1}$.

Suppose $\tau = f$. Then $\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), f) = g_f$ iff $x = (y, z)_{n,m-2}$ for some elements $y, z \in C_{n,m-2}$ such that y originates as $w \in C_k$ with $k \leq n$ and we have $(\mathrm{ind}(w))^{(n)} = z$. Also, $\mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x), f) = g_f$ iff $\varphi_{n,m}(x) = (y', z')_{n+1,m-2}$ for some $y', z' \in C_{n+1,m-2}$ where y' originates as $w' \in C'_k$ with $k' \leq n$ and we have

$(\mathrm{ind}(w'))^{(n+1)} = z'$. Again, the last condition is equivalent to $(\mathrm{ind}(w))^{(n)} = z$, since $\varphi_{n,m}(\mathrm{S}_{n,m-1}(z))$ equals $\mathrm{S}_{n+1,m-1}(\varphi_{n,m-1}(z))$ for any $z \in C_{n,m-1}$.

So we have verified (4-57). As with $\varphi_{n,1}$, we know that $\varphi_{n,m+1}$ is one-to-one, from the uniqueness of normal forms for x, the fact that $C_{n,0}$ is sparse, and also the fact that $\mathrm{S}_{n,m}$ is one-to-one.

Thus we have maintained the induction hypothesis and verified

$$\mathrm{R}^{\varphi}_{n,m+1}(\mathrm{S}_{n,m}(x), \tau) = \mathrm{R}^{\mathrm{S}}_{n+1,m}(\varphi_{n,m}(x), \tau) \tag{4-29}$$

for each $n, m \in \omega$ and each $\tau \in T_{n+1}$. So as we observed with (4-29), it follows from definitions (4-12) of $\varphi_{n,m+1}$ and (4-24) of $\mathrm{S}_{n+1,m}$ that φ_n is an S-algebra homomorphism with $\varphi_n : C_n \xrightarrow{1-1} C_{n+1}$. Now Propn 4.7 is proven. ∎

PROPN 4.10. *In the structure C of Section 4.2, if $\mathrm{N}^C = \{x | \mathrm{S}(x) \leq \omega\}$, then $\langle \mathrm{N}^C, E\rangle \cong \langle \omega, \in \rangle$. Moreover, the element f is (a code for) a one-to-one embedding of u into ω; that is to say, the relation $\{(y,z) | (y,z)Ef\}$ is a well-defined one-to-one function whose range is included in the natural numbers.*

PROOF: Recall that we have made the natural identification of an element in the limit, such as $\langle f \rangle_{\alpha > n}$ for some n, with f itself. Since the connecting morphisms are one-to-one, the direct limit is essentially a union of a chain of S-algebras, so ω in the limit will be a standard ω. Technically, the maps φ_n are not inclusions, so each C_n will have different interpretations of the natural numbers. We defined $\mathrm{R}^{\mathrm{S}}_{n+1,m}(x, \omega) = g_\omega$ iff $x = j^{(n+1)}$ for some $j \leq m$. Then the set of E_n-elements of ω in C_n is just $\{j^{(n)}\}_{j \in \mathrm{N}}$. Each $j^{(n)}$ occurs in stages $C_{n,l}$ for $l \geq j$, where it has exactly j predecessors. So each $j^{(n)}$ has j predecessors in C_n. Since φ_n is a one-to-one homomorphism of S-algebra, we know that $\varphi_n(j^{(n)})$ is $j^{(n+1)}$. So in C, if we let $0' = \mathbf{0}$ and $(j+1)' = j' \vee \mathrm{S}(j')$, then the set of E-elements of ω in C

is just $\{j'\}_{j\in\mathbf{N}}$. Each j' essentially occurs in some stage C_n where it has exactly j predecessors. So each j' has exactly j predecessors in C. Moreover, we have $j'En'$ iff $j \subset n$, i.e. $j' < n'$. So $\langle \mathbf{N}^C, E\rangle \cong \langle \mathbf{N}, \in \rangle$.

Recall that when f originated at some stage in the direct limit system, we chose as its E-elements those (y,z) such that if y originates as w in the direct limit system then $z = (\mathrm{ind}(w))'$. Thus, as w is moved in the development of the direct limit, the associated natural number index remains the same. So f has a well-defined value for each member of C, which is technically of the form $\{\varphi^l_k(w)\}_{l\in\mathbf{N}}$. Since $z = (\mathrm{ind}(w))'$, and ind is one-to-one, we also have that $\{\langle y,z\rangle | (y,z)Ef\}$ is one-to-one, as required for f to code a one-to-one map of u into ω in C. ∎

Now we use Theorem 3.8, which gives us a comprehensive extensional subalgebra of any given S-algebra.

THEOREM 4.11. *There is an* S*-algebra which satisfies*

(i) *extensionality*

(ii) *strictly quantifier-free comprehension with arbitrary parameters*

(iii) *internal countability*

PROOF: (i) Strictly quantifier-free comprehension follows from closure under Gödel-like operations with special constants for C. We will interpret the term e in C as $\langle g_e\rangle_{l>n}$ where n is least such that the term e is in T_n. For each $\langle\varphi^l_k(p)\rangle_{l\geq k}$ in C, we know p occurs first in C_k and the term sets T_n contain standard G-terms with special constants for such objects as p. If τ is a standard G-term with special constants for elements $a_1,\ldots a_j$ of C, let k be least such that C_k contains pre-images $a'_1,\ldots,a'_j$ of all $a_1,\ldots a_j$ under the canonical injections into C. Then τ', the standard G-term formed by replacing the special constants for $a_1,\ldots a_j$ by special constants for $a'_1,\ldots a'_j$, belongs to some T_n, and $C_{n+1} \vDash \forall x[xEg_{\tau'} \leftrightarrow \mathrm{Def}[\tau'](x)]$. Then the

isomorphic image of C_{n+1} under φ_n^l for $l > n$, also satisfies that sentence, as does C_l, by Lemma 4.3. Each generator g_τ, intended to play the role of some Gödel-like construction (which includes the formation of principle ultrafilters), continues to play that role throughout the limit system. Eventually we generate an element to play the role of every standard G-term with special constants for any C_l. Theorem 4.2 tells us that C is closed under Gödel-like operations, which includes formation of $\mathrm{puf}(c_p)$, i.e. $\{y|\, pEy\}^C$. Now, Theorem 2.15 says that the associated membership structure for C satisfies strictly quantifier-free comprehension with arbitrary parameters, since $\mathrm{puf}(c_p) = \{y|\, pEy\}$ is the witness of the only non-trivial comprehension axiom for an atomic formula in which we find a parameter.

(ii) Theorem 3.8 will give an extensional S-subalgebra $B \leq C$ which is comprehensive, i.e. for every $y \in C$ there is an $y' \in B$ with $\mathrm{extn}_B(y') = \mathrm{extn}_B(y)$. Recall that for any S-algebra $C < {}^{\mathbf{N}}2$ and any $y \in {}^{\mathbf{N}}2$, we have defined

$$\mathrm{extn}_C(y) = \{x \in C|\, \mathrm{S}_C(x) \leq y\} \tag{4-62}$$

Theorem 3.8 applies where we have an S-algebra generated by a sparse subalgebra Q which is generated by sups from a tree $G \subset Q$. In the current situation, we have isomorphic images of $C_{n,0}$ included in C. The atoms relative to those subalgebras are the mimimal non-zero Boolean combinations of the generators at each stage. If $C_{\omega,0}$ denotes the union of the isomorphic images of $C_{n,0}$, then those relative atoms form a tree Q which generates $C_{\omega,0}$ by sups. The S-algebra C is $\mathrm{SA}(C_{\omega,0})$. So Theorem 3.8 gives us a subtree which generates by sups a Boolean subalgebra Q' of Q such that $B = \mathrm{SA}(Q')$ is extensional and comprehensive with respect to C.

The comprehensive S-subalgebra B preserves closure under Gödel-like operations, and hence satisfies strictly quantifier-free comprehension with parameters for

$\mathcal{L}(E,=)$. Consider G-terms with special constants for $B = \mathrm{SA}(Q')$. This set of terms is closed under sub-terms and contains every special constant which occurs in these terms. Sparseness of Q' ensures that none of its elements is a sup of finitely many singletons of B. Finally, note that for every τ

$$B \vDash \forall x[xEy' \leftrightarrow \mathrm{Def}[\tau](x)] \tag{4-63}$$

Thus Lemma 4.3 assures us that the representatives in B of the generators associated with G-terms do preserve their roles in the S-subalgebra B of C. So the associated membership structure for B, which is extensional, is also closed under Gödel-like operations, which again includes formation of principle ultrafilters. So B satisfies strictly quantifier-free comprehension with arbitrary parameters.

(iii) The representative ω' in B of the element ω of C contains exactly the same E-elements as ω. The elements are $0' = \mathbf{0}$ and $(j+1)' = j' \vee \mathrm{S}(j')$ for all $j \in \omega$. These interpreted natural numbers are exactly the E-elements of ω in C. Consider the representative $f' \in B$ of the element $f \in C$. Note that a pair $(y, \mathrm{ind}(w))'$ in C is omitted from B only if y is omitted from B. So the E-elements of f' in B are the pairs $x = (y,z) \in B$ such that y originates as w and $z = (\mathrm{ind}(w))'$. Thus the domain of the function (code) f' is all of B. That is to say, the structure B satisfies $f' : u \xrightarrow{1-1} \omega'$. ∎

CHAPTER 5

CONCLUSION

The immediate conclusion is that we have constructed a model with these features

(i) The model satisfies a comprehension axiom scheme which applies to almost all quantifier-free formulas with arbitrary parameters, and each individual comprehension axiom in this scheme is identical to the simple form of the original set theory.

(ii) Extensionality

(iii) Power set closure

(iv) The model satisfies the statement "ω exists"

(v) The model satisfies an axiom of internal countability for every set in the universe.

The model is closed under many of the operations which Gödel used in the construction of L. In fact some of these operations are strengthened in our model, such as absolute complement and existence of unrestricted membership and identity relations.

This method of construction shows considerable flexibility beyond its basic purpose, which is to prove consistency of extensionality with the comprehension scheme for strictly quantifier-free formulas without parameters. That requires only a structure with definable elements. In closing under the Gödel-like operations, we begin with a basic structure which contains ω and the internal embedding $f : u \xrightarrow{1-1} \omega$. The element ω is not definable by quantifier-free formulas, and f will likely not be definable in any substantially restrictive sense within the model itself.

This construction exploits the Boolean structure of the intermediate stages in three ways. First, it uses the order of the Boolean algebra to allow the combination of many set membership decisions in the single step of defining $S(x)$ for a given x, while at the same time it allows most of those decisions to be changeable. Second, it uses the fact that countable Boolean algebras are projective to allow extraction of a significant extensional substructure. Third, it uses the order of the Boolean algebra to achieve a simple representaton of the power set function.

One natural question for investigation is whether an arbitrary countable model $\mathcal{A}$ of ZF could be extended to an S-algebra closed under Gödel-like operations. This seems very likely with little modification of the procedure, since, as noted above, we have closed a structure under Gödel-like operations even though the element f had no restrictive definability properties. The essential requirement, as with having ω and f in a structure, is just that we have at hand sufficient information about the given model. This would be the case since we would be given the model $\mathcal{A}$ itself, and its natural extension would add no new elements to any member of $\mathcal{A}$.

Another aspect of interest in this construction is its natural generalization of the notion of "cumulative hierarchy" beyond well-founded structures of sets. So long as the structure of sets is an atomic Boolean algebra, and the model in question is countable (in the real world), it is the S-algebra generated by ranks with respect to a sparse Boolean subalgebra. The partition of the universe of sets with this rank hierarchy expresses the idea that we are given, for mathematical construction, the empty set and also the universe and the primitive relations of our mathematical language. Specifically, for our constructions we begin with the relations of membership and identity and the elements ω and $f : u \xrightarrow{1-1} \omega$.

This picture of a non-well-founded universe is quite distinct from the picture suggested by an anti-foundation axiom such as proposed by I. Malitz [16]. That axiom

declares the existence of a (unique) set associated with each directed graph of a certain type, including many graphs with cycles which correspond to non-well-founded constructions. Our structure maintains well-foundedness relative to the primitives of our language and other relations directly definable from them.

We must admit non-well-foundedness where it is essential to a domain of investigation. For example, where true universality of a proposition is of interest, i.e. where we want to know its status in "the real world", we accept as fact that the universe, like anything else, is a member of the universe. As another example, one might want to represent the world view of an individual in order to apply mathematical logic in a social science, or to program computation systems for more adaptability to their users. In such cases, we accept as fact that each of us is aware of himself or herself as a part of the world envisioned in his or her own mind. Considering the following quite different appearance of non-well-foundedness.

If there is an object x such that $x = \{x\}$, and one uses the traditional Weiner-Kuratowski codes for ordered pairs, then $x = (x, x)$. If we code (x, x, x) as $((x, x), x)$, and similarly for other ordered n-tuples, then $x = (x, x) = (x, x, \ldots, x)$ and we have $xE(\bigcap\{u^n\}_n)$. This phenemenon does not occur in the method of this dissertation. The directed graph naturally associated with any of our ordered pairs is a finite binary tree.

Due to the formal nature of the construction, and the significance of countability in it, another question for further investigation is whether this method can produce a concrete model of the lambda calculus. This is especially desirable since one of the motivations we mentioned for set theory with a universal set u was that the universe is effectively decidable, and we naturally want a model in which all effectively decidable functions and relations are representable.

In analogy to second order arithmetic, it would be desirable to go beyond the lambda calculus, which is associated with a comprehension scheme for quantifier-free formulas, to some axioms fairly close to the theory Σ_1-CA in the program of reverse mathematics.

How much strength can a foundational theory of mathematics have, if the theory uses a limitation of complexity doctrine based on strictly quantifier-free formulas? That is, how much of algebraic and analytic construction can be done in a model of a comprehension scheme limited to simple formulas, if we evaluate a formulas complexity as the number of quantifiers in its logical equivalents which have no variable occurring more than once in any atomic subformula?

Consider what the formal rules of existential quantification require of an interpretation of quantified formulas. One rule is that for every formula $\mathrm{A}(x,y)$, the statement $\vdash \mathrm{A}(w,y) \longrightarrow \exists x\, \mathrm{A}(x,y)$ is a theorem. This is an axiom in some formulations of logic. So given a relation a, if $\mathcal{Q}(a)$ interprets $\exists x\, \mathrm{A}(x,y)$, then $\mathcal{Q}(a)$ must include the range of a.

The other rule, existential elimination, is not so simple. It says that for any formulas $\mathrm{A}(x,y)$ and $\mathrm{B}(y)$, if $\vdash \mathrm{A}(w,y) \longrightarrow \mathrm{B}(y)$ is provable, then we can deduce $\vdash \exists x\, \mathrm{A}(x,y) \longrightarrow \mathrm{B}(y)$, provided w does not occur in $\mathrm{B}(y)$. So, if a is $\mathrm{A}(x,y)^{\mathcal{M}}$, the interpretation in a model $\mathcal{M}$ of $\mathrm{A}(x,y)$, then $[\exists x\, \mathrm{A}(x,y)]^{\mathcal{M}}$ must respect every upper bound b on the (hypothetical) range of a, so long as b corresponds to a formula $\mathrm{B}(y)$.

Thus we arrive at the following suggestion for a significant weak replacement axiom.

DEFN 5.1. In the language of S-algebras with an operator $\mathcal{Q}(\cdot)$ and a predicate $D(\cdot)$, the axiom of *weak replacement*, or *uniform separation*, is this pair of axioms

$$\forall a \forall y\, [[\exists x(x,y) E a] \rightarrow y E \mathcal{Q}(a)] \tag{5-1}$$

$$\forall a \forall b\,[D(b) \to [[\exists x(x,y)Ea] \to yEb] \to \forall x\,[xE\mathcal{Q}(a) \to xEb]]]$$

together with a finite set of axioms which say that $D(\cdot)$ is interpreted as a Boolean subalgebra of a given model. A model $\mathcal{A}$ of the theory is a structure $\langle A, B, \wedge, \vee, \sim, \mathrm{S}\rangle$ and the Boolean subalgebra $B = D^{\mathcal{M}}$ in A is referred to as the subalgebra of *definable* elements of $\mathcal{A}$.

Note that the relation a need not be single-valued, since the definition is to express a limitation of complexity doctrine, so the size of a set is irrelevant. It is easy to see that $\mathcal{Q}(a) = u$ for all $a \neq \mathbf{0}$, with $\mathcal{Q}(\mathbf{0}) = \mathbf{0}$ defines a trivial replacement operator which satisfies the formulas of (5-1). So it is the upper bounds restricted by a fairly broad notion of definability which may make such an operator useful. Note that $\vdash \mathrm{A}(w,y) \longrightarrow \exists x\,\mathrm{A}(x,y)$ does not require w to be definable, so we do not need to change our interpretation of that theorem in view of our understanding of $\exists$-introduction.

Let us examine the obvious attempt to derive Russell's contradiction with these additional axioms about quantification. Suppose e and i are (elements which code) the unrestricted membership and identity relations. The analog of Russell's class is $r = \mathcal{Q}((\sim e) \wedge i)$. We can prove that rEr. Suppose $r \not E r$. Then $(r,r)E \sim e$. For all x, we have $(x,x)Ei$. Thus,

$$(5\text{-}2) \qquad (r,r)Ei \Longrightarrow (r,r)E((\sim e) \wedge i) \Longrightarrow rE\mathcal{Q}((\sim e) \wedge i) \Leftrightarrow rEr$$

The conclusion contradicts the assumption that $r \not E r$. Thus rEr.

Weak replacement preempts the conflicting argument. We have rEr, which is to say $rE\mathcal{Q}((\sim e) \wedge i)$. For all a and for all $b \in D^{\mathcal{M}}$, we have

$$(5\text{-}3) \qquad \vdash [\forall y\,[\exists x(x,y)Ea] \to yEb] \to \forall y\,[yE\mathcal{Q}(a) \to yEb]$$

So, in particular, for Russell's set $r = \mathcal{Q}((\sim e) \wedge i)$ and any $b \in D^{\mathcal{M}}$, we have

(5-4) $\quad [\forall y\,[\exists x\,(x,y)E((\sim e) \wedge i)] \rightarrow yEb] \rightarrow \forall y\,[yE\mathcal{Q}((\sim e) \wedge i) \rightarrow yEb]$

Now since rEr, whence $r \not\in \sim \mathrm{S}(r)$, we take $b = \sim \mathrm{S}(r)$, and

(5-5) $$\begin{aligned} &\neg\forall y\,[yE\mathcal{Q}((\sim e) \wedge i) \rightarrow yE \sim \mathrm{S}(r)] \\ &\quad\Longrightarrow \neg\forall y\,[\exists x\,(x,y)E(\sim e \wedge i) \rightarrow yE \sim \mathrm{S}(r)] \\ &\quad\Longrightarrow \exists y\,\exists x[(x,y)E(\sim e \wedge i) \wedge yES(r)] \\ &\quad\Longrightarrow \exists x\,(x,r)E(\sim e \wedge i) \\ &\quad\Longrightarrow (r,r)E \sim e \\ &\quad\Longrightarrow r \not\in r \end{aligned}$$

But this only proves $\neg D(\sim \mathrm{S}(r))$, or $\sim \mathrm{S}(r) \not\in D^{\mathcal{M}}$. It is natural to require in addition that $D^{\mathcal{M}}$ is closed under singletons, since for any $p \in D$ the formula $x = p$ defines $\mathrm{S}(p)$. Even so, Russell's argument only proves that Russell's class is not definable.

If we require D to be closed under S, the Boolean operations and quantifier-free comprehension, and we require $B = D^{\mathcal{M}}$ to contain e, it follows that the unrestricted identity relation cannot be in $D^{\mathcal{M}}$. Since $\cdot = \cdot$ is definable from $\cdot E \cdot$ with a single (weak) universal quantifier, if we define the element i from e and $\mathcal{Q}$ in the natural way, then i must be a restriction of the identity relation. (Just as a Σ_1 formula has an interpretation larger that one would expect, a universal formula has an interpretation smaller thn one would expect.) So the weak replacement operator is indeed distinct from the range operator.

While the restriction of the element i is a limitation, it does extend the analogy with second order arithmetic. In the arithmetic context, equality of natural numbers

is an effectively decidable relation, but equality of a pair of reals is a Π_1 relation, relative to those reals, defined by extensional equality of the sets which are (codes for) a given pair of reals.

Finally we conclude that the notion of strictly quantifier-free formulas and our method for construction of a model which satisfies strictly quantifier-free comprehension constitute a step toward the goal of resolving set theoretic paradoxes by a limitation of complexity doctrine, and at the same time preserve enough axiomatic strength to support substantial mathematics.

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BIOGRAPHICAL SKETCH

After completion of the B.S. degree in mathematics and M.S. degree in topology at the University of Illinois in 1971, Mr. Jamieson pursued an interest in application of his mathematical training. He studied in the Electrical Engineering Department at the University of Illinois, initially as a National Science Foundation fellow. In those studies he developed an interest in potential applications of mathematical logic in computer science, computational psychology and the social sciences. He returned to the study of mathematics at the Pennsylvania State University with a concentration in logic. His professors there, primarily Stephen Simpson, exposed him to foundational work in mathematics which contributed to his view of the relationship between mathematics and his preferred areas of application. It was also there that he met Rick Smith, the colleague who was later to supervise his doctoral work in the University of Florida Department of Mathematics, where he continued to advance the formulation of a foundational theory to support mathematics in application to computer science, computational psychology and the social sciences.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Rick L. Smith, Chairman
Associate Professor of Mathematics

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William J. Mitchell
Professor of Mathematics

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Jean A. Larson
Professor of Mathematics

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Douglas A. Cenzer
Professor of Mathematics

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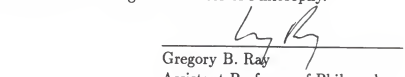

Gregory B. Ray
Assistant Professor of Philosophy

This dissertation is submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Dean, Graduate School